CHOOSING A RANKING OF ALTERNATIVES IN A TWO-STAGE GROUP DECISION PROCEDURE WITH INDIVIDUAL LINGUISTIC ASSESSMENTS

José Luis García-Lapresta, Bonifacio Llamazares

In this paper we provide a two-stage group decision-making procedure for ranking a great number of alternatives. Since individuals usually have inconsistencies in the pairwise comparison of alternatives, we allow agents to assess alternatives one by one using linguistic labels. This information is processed by using aggregation functions, each one generating a complete preorder as a partial decision group outcome. In order to choose the final ranking of alternatives we consider an aggregation rule based on the classical Borda count, in which weights over partial outcomes are considered.

Keywords: *group decision-making, linguistic labels, Borda count, ordered semigroups, complete preorders, trapezoidal fuzzy numbers, approval voting.*

1. INTRODUCTION

In some multiperson decision problems is necessary to choose a ranking of alternatives taking into account the individual opinions about a large set of alternatives. Because agents usually have difficulties to compare coherently all the pairs of alternatives (cycles in preferences easily appear), it is common to allow them to assess the alternatives one by one. Moreover, linguistic assessments are allowed because agents generally are not able to provide exact numerical values.

In the ordinary decision framework, approval voting (see Brams and Fishburn (1978, 1983)) is one of the more straightforward methods for choosing one or several alternatives, especially when there are a great number of them. This procedure only requires that agents approve of as many alternatives as they wish. Then, the alternative(s) with the greatest number of votes is (are) selected as the best. Ylmaz (1999) considers a substitute voting method with three categories rather than two. However, human beings usually use more than three kinds of linguistic assessments. These reasons lead us to propose a general framework where individuals can assess the alternatives by means of a general set of linguistic labels.

There is a wide class of aggregation functions which rank alternatives taking into account individual linguistic assessments. In this paper we provide a two-stage group decision-making procedure for ranking alternatives. An initial contribution developed under this approach can be found in García-Lapresta (2003).

In the first-stage we consider several aggregation functions that generate complete preorders on the set of alternatives. For ranking the assessed alternatives these functions need to add up the individual assessments (linguistic labels) and to order the obtained sums. For this purpose we provide a general framework, similar to one given by García-Lapresta (forthcoming), based on totally ordered commutative semigroups generated by the original linguistic labels.

In order to show the first stage of the group decision procedure we present two fuzzy generalizations of the Borda count, related to García-Lapresta, Lazzari and Martínez-Panero (2001), and a fuzzy generalization of approval voting. With the information provided by the agents in the classification of the alternatives, the mentioned aggregation functions consider that each agent assigns an element of the semigroup generated by the linguistic labels to every alternative: the sum of the fuzzy qualifications corresponding to the alternatives that are evaluated worse (or equal) than it, for the fuzzy Borda counts; and the linguistic label corresponding to the evaluated alternative, for the fuzzy approval voting. In all the cases the global qualification of an alternative is the addition of the individual ones. Taking into account the ordering associated with the totally ordered semigroup, we obtain a complete preorder on the set of alternatives for each aggregation function.

Since different aggregation functions could rank the alternatives in a different manner, we have introduced a second stage decision procedure in order to choose the final ranking of the alternatives. For this reason we use an aggregation rule over the above complete preorders. This rule is based on the classical Borda count, in which weights over partial outcomes (obtained in the first stage) are considered. On classical Borda count and two generalizations in a fuzzy pairwise comparison framework, see García-Lapresta, Lazzari and Martínez-Panero (2001) and García-Lapresta and Martínez-Panero (2002).

By simplicity, in the examples contained in this paper we suppose that agents evaluate alternatives by using five categories widely used in practice: very good, good, medium, bad and very bad. Consequently, agents classify the alternatives according to this class of linguistic terms. In order to add up the corresponding assessments, each linguistic label is represented by means of a trapezoidal fuzzy number. The obtained global qualification of an alternative is the sum of all the individual ones. For comparing the reached results for alternatives, an ordering in the set of trapezoidal fuzzy numbers is needed. In the examples we have considered one given by Delgado, Vila and Voxman (1998). Through both elements, fuzzy numerical representation of the linguistic labels (with the usual sum) and the mentioned ordering, it is easy to put in practice the aggregation functions.

2. SORTING ALTERNATIVES BY MEANS OF LINGUISTIC LABELS

Suppose m agents, $m \geq 3$, who have to evaluate alternatives of $X = \{x_1, \ldots, x_n\}$, $n \geq 3$, by means of a set of linguistic labels $L = \{l_0, l_1, \ldots, l_s\}$, $s \geq 2$, ranked by a linear order: $l_0 < l_1 < \cdots < l_s$. Suppose the number of labels, $s+1$, is odd; consequently, $l_{s/2}$ is the central label and the rest of labels are defined around it symmetrically.

Linguistic labels can be represented mainly by real numbers, intervals and triangular and trapezoidal fuzzy numbers. Particular representations of linguistic labels by means of fuzzy numbers can be found in Zadeh (1975), Marimin, Umano, Hatono and Tamura (1998), Herrera and Herrera-Viedma (2000) and García-Lapresta, Lazzari and Martínez-Panero (2001), among others. We also note that linguistic labels can be managed symbolically by means of the linguistic OWA operators introduced in Herrera, Herrera-Viedma and Verdegay (1996).

We present now a general framework for considering linguistic evaluations of alternatives similar to one given by García-Lapresta (forthcoming).

Let $(\langle L \rangle, +)$ be the commutative semigroup generated by L and an associative and commutative operation $+$ on L:

1. $L \subset \langle L \rangle$.
2. $l + l' \in \langle L \rangle$, for all $l, l' \in \langle L \rangle$.
3. $l + (l' + l'') = (l + l') + l''$, for all $l, l', l'' \in \langle L \rangle$.
4. $l + l' = l' + l$, for all $l, l' \in \langle L \rangle$.

We also consider a total order $\leq$ on $\langle L \rangle$ compatible with the original linear order on L:

5. $l \leq l$, for all $l \in \langle L \rangle$.
6. $(l \leq l'$ and $l' \leq l) \Rightarrow l = l'$, for all $l, l' \in \langle L \rangle$.
7. $(l \leq l'$ and $l' \leq l'') \Rightarrow l \leq l''$, for all $l, l', l'' \in \langle L \rangle$.
8. $l \leq l'$ or $l' \leq l$, for all $l, l' \in \langle L \rangle$.

9. $l_0 < l_1 < \cdots < l_s$, where $<$ is the strict order associated with $\leq$ $\left(l < l' \text{ if } l \leq l' \text{ and } l \neq l', \text{ for all } l, l' \in \langle L \rangle \right)$.

Moreover, we suppose compatibility between $+$ and $\leq$:

10. $l \leq l' \Rightarrow l + l'' \leq l' + l''$, for all $l, l', l'' \in \langle L \rangle$.

Then, $\left(\langle L \rangle, +, \leq \right)$ is a totally ordered commutative semigroup.

Suppose agents assess alternatives by means of evaluation functions

$$
\begin{aligned}
v_k :\ & X \longrightarrow L \\
& x_i \longrightarrow v_k(x_i),
\end{aligned}
$$

$k = 1, \ldots, m$, where $v_k(x_i)$ is the evaluation of x_i by the agent k. A profile is a vector $(v_1, \ldots, v_m)$ of individual evaluation functions and ν is the set of profiles. Profiles can be expressed by means of linguistic matrices

$$
\begin{pmatrix}
v_1(x_1) & v_1(x_2) & \cdots & v_1(x_n) \\
v_2(x_1) & v_2(x_2) & \cdots & v_2(x_n) \\
\cdots & \cdots & \cdots & \cdots \\
v_m(x_1) & v_m(x_2) & \cdots & v_m(x_n)
\end{pmatrix}.
$$

These matrices provide a classification of alternatives with respect to the linguistic labels: for each agent k and each linguistic label l_h we have

$$
C_k(l_h) = v_k^{-1}\left(\{ l_h \} \right) = \{ x_i \in X \mid v_k(x_i) = l_h \} ,
$$

the set of alternatives which agent k evaluates with the linguistic label l_h.

Now let $R(X)$ be the set of complete preorders on X, i.e., ordinary binary relations R on X which are complete and transitive:

1. $x_i R x_j$ or $x_j R x_i$, for every $x_i, x_j \in X$.
2. $x_i R x_k$ whenever $x_i R x_j$ and $x_j R x_k$, for every $x_i, x_j, x_k \in X$.

$R \in R(X)$ is understood as a weak preference relation: $x_i R x_j$ means that x_i is at least as good as x_j. The strong preference relation P associated with R is defined by $x_i P x_j$ if not $x_j R x_i$ and means that x_i is better than x_j. The indifference relation I associated with R is defined by $x_i I x_j$ if $x_i R x_j$ and $x_j R x_i$, and means that x_i is indifferent to x_j.

We note that from this construction we have that P is asymmetric and transitive, and I is reflexive, symmetric and transitive:

1. If $x_i P x_j$, then not $x_j P x_i$, for every $x_i, x_j \in X$.
2. $x_i P x_k$, whenever $x_i P x_j$ and $x_j P x_k$, for every $x_i, x_j, x_k \in X$.
3. $x_i I x_i$, for every $x_i \in X$.
4. If $x_i I x_j$, then $x_j I x_i$, for every $x_i, x_j \in X$.
5. $x_i I x_k$, whenever $x_i I x_j$ and $x_j I x_k$, for every $x_i, x_j, x_k \in X$.

Example 1. Suppose 5 agents who have to assess the alternatives of $X = \{x_1, \ldots, x_8\}$ by using the linguistic labels $L = \{l_0, l_1, l_2, l_3, l_4\}$ whose meaning and their associated trapezoidal fuzzy numbers are given in the semantics of Table 1.

Table 1. Semantics of the linguistic labels

Label	Meaning	Trapezoidal fuzzy number
l_0	very bad	$(0, 0, 0, 0.2)$
l_1	bad	$(0, 0.125, 0.275, 0.4)$
l_2	medium	$(0.2, 0.4, 0.6, 0.8)$
l_3	good	$(0.6, 0.725, 0.875, 1)$
l_4	very good	$(0.8, 1, 1, 1)$

Suppose the profile given by the following matrix

$$\begin{pmatrix} l_1 & l_4 & l_4 & l_3 & l_2 & l_3 & l_1 & l_0 \\ l_3 & l_2 & l_3 & l_4 & l_1 & l_0 & l_2 & l_4 \\ l_1 & l_3 & l_1 & l_3 & l_4 & l_0 & l_4 & l_0 \\ l_2 & l_4 & l_2 & l_3 & l_4 & l_1 & l_0 & l_0 \\ l_2 & l_3 & l_1 & l_3 & l_4 & l_1 & l_0 & l_2 \end{pmatrix}$$

which provides the following classification of alternatives

Table 2. Individual classification of alternatives

	k = 1	k = 2	k = 3	k = 4	k = 5
$C_k(l_4)$	x_2, x_3	x_4, x_8	x_5, x_7	x_2, x_5	x_5
$C_k(l_3)$	x_4, x_6	x_1, x_3	x_2, x_4	x_4	x_2, x_4
$C_k(l_2)$	x_5	x_2, x_7		x_1, x_3	x_1, x_8
$C_k(l_1)$	x_1, x_7	x_5	x_1, x_3	x_6	x_3, x_6
$C_k(l_0)$	x_8	x_6	x_6, x_8	x_7, x_8	x_7

3. AGGREGATION FUNCTIONS

An aggregation function $A : \nu \to R(X)$ assigns a complete preorder R on X to each profile $(v_1,\ldots,v_m)$, $R = A(v_1,\ldots,v_m)$. Given r aggregation functions $A_1,\ldots,A_r$, with r ≥ 2, we will consider the aggregator $\mathbf{A} : \nu \to R(X)^r$ which assigns r complete preorders $R_1,\ldots,R_r$ to each profile $(v_1,\ldots,v_m)$, according to the aggregation functions, i.e., $R_i = A_i(v_1,\ldots,v_m)$.

As an example we will consider three concrete aggregation functions based on two well-known group decision procedures: the Borda count and the approval voting. First of all we present the related crisp procedures.

The classical Borda count supposes that individuals rank the alternatives by means of linear orders (complete preorders such that no different alternatives are indifferent). Every agent assigns to each alternative a mark: the number of alternatives ranked worse than it. Then the winner alternative(s) is (are) that who obtain the highest score. When individuals rank alternatives by means of a complete preorder (indifferences could appear) instead of a linear order, several generalizations of the classical Borda count have been considered in the literature. We have taken into account two of them, but adapting the notation to our framework. The first Borda count has been used by Gärdenfors (1973) and Nitzan and Rubinstein (1981); the second one is equivalent to one given by Black (1976).

1. For every $h \in \{1,\ldots,s\}$, if $x_i \in C_k(l_h)$, then the agent k assigns to x_i the score

$$\sum_{h'=0}^{h-1} \text{card } C_k(l_{h'}) .$$

If $x_i \in C_k(l_0)$, then its score is 0.

Similarly to the classical Borda count, each agent gives a mark to each alternative: the number of alternatives worse than it. Taking into account the individual marks, the total score of an alternative is defined by the addition of the individual marks. Then the winner alternative(s) is (are) that which obtain the highest total score.

2. For every $h \in \{0,\ldots,s\}$, if $x_i \in C_k(l_h)$, then the agent k assigns to x_i the score

$$\sum_{h'=0}^{h} \text{card } C_k(l_{h'}) .$$

Thus, each agent gives a mark to each alternative: the number of alternatives worse than it or indifferent to it. Again taking into account the individual marks, the total score of an alternative is defined by the addition of the individual marks. Then the winner alternative(s) is (are) that which obtain the highest total score.

Approval voting supposes that each agent chooses the good alternatives, assigning 1 point to each one, and giving 0 points to the others. If we consider that agents qualify as good alternatives those sorted with labels greater than the central label, then we can define this crisp group decision procedure in the following way.

3. If $x_i \in C_k(l_h)$, then the agent k assigns 1 to this alternative whenever $h > \frac{s}{2}$ and 0 otherwise. Again taking into account the individual marks, the total score of an alternative is defined by the addition of the individual marks. Then the winner alternative(s) is (are) that which obtain the highest total score.

Now we present the aggregation functions based on the previous crisp group decision procedures. In this case, each agent k assigns an element of the commutative semigroup generated by L to each alternative:

$$\begin{aligned} s_k : \quad X &\longrightarrow \langle L \rangle \\ x_i &\longrightarrow s_k(x_i). \end{aligned}$$

1. A_1 is based on the first version of the classical Borda count, but considering indirect fuzzy preferences between the evaluated alternative and those worse than it. For every $h \in \{1,\ldots,s\}$, if $x_i \in C_k(l_h)$, then the agent k assigns to x_i the fuzzy score
$$s_k^1(x_i) = \sum_{h'=0}^{h-1} \text{card } C_k(l_{h'}) \cdot l_{h-h'} .$$
If $x_i \in C_k(l_0)$, then its score is l_0.
Taking into account the individual marks, A_1 assigns the complete preorder R_1 defined by
$$x_i R_1 x_j \Leftrightarrow \sum_{k=1}^{m} s_k^1(x_i) \geq \sum_{k=1}^{m} s_k^1(x_j) .$$
Then the winner alternative(s) is (are) that who obtain the highest total fuzzy score according to $(\langle L \rangle,+,\leq)$.

2. A_2 is based on the second version of the classical Borda count, again considering indirect fuzzy preferences between the evaluated alternative and those worse than or indifferent to it. For every $h \in \{0,\ldots,s\}$, if $x_i \in C_k(l_h)$, then the agent k assigns to x_i the fuzzy score
$$s_k^2(x_i) = \sum_{h'=0}^{h} \text{card } C_k(l_{h'}) \cdot l_{h-h'} .$$
Now A_2 assigns the complete preorder R_2 defined by
$$x_i R_2 x_j \Leftrightarrow \sum_{k=1}^{m} s_k^2(x_i) \geq \sum_{k=1}^{m} s_k^2(x_j) .$$
Again, the winner alternative(s) is (are) that who obtain the highest total fuzzy score according to $(\langle L \rangle,+,\leq)$.

3. A_3 is a linguistic generalization of the approval voting. For every $h \in \{0,\ldots,s\}$, if $x_i \in C_k(l_h)$, then the agent k assigns to x_i the fuzzy

score $s_k^3(x_i) = l_h$. Similarly to the previous cases, A_3 assigns the complete preorder R_3 defined by

$$x_i R_3 x_j \Leftrightarrow \sum_{k=1}^{m} s_k^3(x_i) \geq \sum_{k=1}^{m} s_k^3(x_j) .$$

Again, the winner alternative(s) is (are) that which obtain the highest total fuzzy score according to $(\langle L \rangle, +, \leq)$.

Example 2. Consider the semantics and the individual assessments provided in Example 1. In order to obtain the complete preorders associated with the three aggregation functions, we use the usual addition of trapezoidal fuzzy numbers and the ordering given by Delgado, Vila and Voxman (1998).

Given two trapezoidal fuzzy numbers (a,b,c,d), (a',b',c',d'):

- $(a,b,c,d) + (a',b',c',d') = (a+a', b+b', c+c', d+d')$.
- $(a,b,c,d) \leq (a',b',c',d') \Leftrightarrow$

$$\Leftrightarrow \begin{cases} a+2b+2c+d < a'+2b'+2c'+d' \\ \text{or} \\ a+2b+2c+d = a'+2b'+2c'+d' \ \text{ and } \ -a-2b+2c+d \leq - \ a'-2b'+2c'+d'. \end{cases}$$

Now we present the individual and collective fuzzy scores joint with the complete preorders associated with the three aggregation functions.

1. A_1 gives the following fuzzy scores:

Table 3. Individual and collective scores provided by A_1

	$s_k^1(x_1)$	$s_k^1(x_2)$
$k=1$	$(0, 0.125, 0.275, 0.4)$	$(2.2, 3.1, 3.9, 4.6)$
$k=2$	$(0.8, 1.375, 2.025, 2.6)$	$(0.2, 0.525, 0.875, 1.2)$
$k=3$	$(0, 0.25, 0.55, 0.8)$	$(1.6, 2.25, 2.95, 3.6)$
$k=4$	$(0.4, 0.925, 1.475, 2)$	$(2.6, 3.65, 4.35, 5)$
$k=5$	$(0.2, 0.65, 1.15, 1.6)$	$(1, 1.775, 2.625, 3.4)$
$\sum_{k=1}^{5} s_k^1(x_i)$	$(1.4, 3.325, 5.475, 7.4)$	$(7.6, 11.3, 14.7, 17.8)$

	$s_k^1(x_3)$	$s_k^1(x_4)$
$k=1$	$(2.2, 3.1, 3.9, 4.6)$	$(1, 1.65, 2.35, 3)$
$k=2$	$(0.8, 1.375, 2.025, 2.6)$	$(1.8, 2.775, 3.625, 4.4)$
$k=3$	$(0, 0.25, 0.55, 0.8)$	$(1.6, 2.25, 2.95, 3.6)$
$k=4$	$(0.4, 0.925, 1.475, 2)$	$(1.4, 2.1, 2.9, 3.6)$
$k=5$	$(0, 0.125, 0.275, 0.4)$	$(1, 1.775, 2.625, 3.4)$
$\sum_{k=1}^{5} s_k^1(x_i)$	$(3.4, 5.775, 8.225, 10.4)$	$(6.8, 10.55, 14.45, 18)$

	$s_k^1(x_5)$	$s_k^1(x_6)$
$k=1$	$(0.2, 0.65, 1.15, 1.6)$	$(1, 1.65, 2.35, 3)$
$k=2$	$(0, 0.125, 0.275, 0.4)$	$(0, 0, 0, 0.2)$
$k=3$	$(2.8, 3.7, 4.3, 4.8)$	$(0, 0, 0, 0.2)$
$k=4$	$(2.6, 3.65, 4.35, 5)$	$(0, 0.25, 0.55, 0.8)$
$k=5$	$(2.4, 3.5, 4.5, 5.4)$	$(0, 0.125, 0.275, 0.4)$
$\sum_{k=1}^{5} s_k^1(x_i)$	$(8, 11.625, 14.575, 17.2)$	$(1, 2.025, 3.175, 4.6)$

	$s_k^1(x_7)$	$s_k^1(x_8)$
$k=1$	$(0, 0.125, 0.275, 0.4)$	$(0, 0, 0, 0.2)$
$k=2$	$(0.2, 0.525, 0.875, 1.2)$	$(1.8, 2.775, 3.625, 4.4)$
$k=3$	$(2.8, 3.7, 4.3, 4.8)$	$(0, 0, 0, 0.2)$
$k=4$	$(0, 0, 0, 0.2)$	$(0, 0, 0, 0.2)$
$k=5$	$(0, 0, 0, 0.2)$	$(0.2, 0.65, 1.15, 1.6)$
$\sum_{k=1}^{5} s_k^1(x_i)$	$(3, 4.35, 5.45, 6.8)$	$(2, 3.425, 4.775, 6.6)$

Therefore, A_1 provides the ranking (linear order):

$$x_5 \ P_1 \ x_2 \ P_1 \ x_4 \ P_1 \ x_3 \ P_1 \ x_7 \ P_1 \ x_1 \ P_1 \ x_8 \ P_1 \ x_6 .$$

2. A_2 gives the following fuzzy scores:

Table 4. Individual and collective scores provided by A_2

	$s_k^2(x_1)$	$s_k^2(x_2)$
$k=1$	$(0, 0.125, 0.275, 0.8)$	$(2.2, 3.1, 3.9, 5)$
$k=2$	$(0.8, 1.375, 2.025, 3)$	$(0.2, 0.525, 0.875, 1.6)$
$k=3$	$(0, 0.25, 0.55, 1.2)$	$(1.6, 2.25, 2.95, 4)$
$k=4$	$(0.4, 0.925, 1.475, 2.4)$	$(2.6, 3.65, 4.35, 5.4)$
$k=5$	$(0.2, 0.65, 1.15, 2)$	$(1, 1.775, 2.625, 3.8)$
$\sum_{k=1}^{5} s_k^2(x_i)$	$(1.4, 3.325, 5.475, 9.4)$	$(7.6, 11.3, 14.7, 19.8)$

	$s_k^2(x_3)$	$s_k^2(x_4)$
$k=1$	$(2.2, 3.1, 3.9, 5)$	$(1, 1.65, 2.35, 3.4)$
$k=2$	$(0.8, 1.375, 2.025, 3)$	$(1.8, 2.775, 3.625, 4.8)$
$k=3$	$(0, 0.25, 0.55, 1.2)$	$(1.6, 2.25, 2.95, 4)$
$k=4$	$(0.4, 0.925, 1.475, 2.4)$	$(1.4, 2.1, 2.9, 3.8)$
$k=5$	$(0, 0.125, 0.275, 0.8)$	$(1, 1.775, 2.625, 3.8)$
$\sum_{k=1}^{5} s_k^2(x_i)$	$(3.4, 5.775, 8.225, 12.4)$	$(6.8, 10.55, 14.45, 19.8)$

	$s_k^2(x_5)$	$s_k^2(x_6)$
$k=1$	$(0.2, 0.65, 1.15, 1.8)$	$(1, 1.65, 2.35, 3.4)$
$k=2$	$(0, 0.125, 0.275, 0.6)$	$(0, 0, 0, 0.2)$
$k=3$	$(2.8, 3.7, 4.3, 5.2)$	$(0, 0, 0, 0.4)$
$k=4$	$(2.6, 3.65, 4.35, 5.4)$	$(0, 0.25, 0.55, 1)$
$k=5$	$(2.4, 3.5, 4.5, 5.6)$	$(0, 0.125, 0.275, 0.8)$
$\sum_{k=1}^{5} s_k^2(x_i)$	$(8, 11.625, 14.575, 18.6)$	$(1, 2.025, 3.175, 5.8)$

	$s_k^2(x_7)$	$s_k^2(x_8)$
$k=1$	$(0, 0.125, 0.275, 0.8)$	$(0, 0, 0, 0.2)$
$k=2$	$(0.2, 0.525, 0.875, 1.6)$	$(1.8, 2.775, 3.625, 4.8)$
$k=3$	$(2.8, 3.7, 4.3, 5.2)$	$(0, 0, 0, 0.4)$
$k=4$	$(0, 0, 0, 0.4)$	$(0, 0, 0, 0.4)$
$k=5$	$(0, 0, 0, 0.2)$	$(0.2, 0.65, 1.15, 2)$
$\sum_{k=1}^{5} s_k^2(x_i)$	$(3, 4.35, 5.45, 8.2)$	$(2, 3.425, 4.775, 7.8)$

Consequently, A_2 gives the ranking (linear order):

$$x_2 \; P_2 \; x_5 \; P_2 \; x_4 \; P_2 \; x_3 \; P_2 \; x_7 \; P_2 \; x_1 \; P_2 \; x_8 \; P_2 \; x_6.$$

3. A_3 gives the following fuzzy scores:

Table 5. Collective scores provided by A_3

	$\sum_{k=1}^{5} s_k^3(x_i)$		$\sum_{k=1}^{5} s_k^3(x_i)$
$i=1$	$(1, 1.775, 2.625, 3.4)$	$i=5$	$(2.6, 3.525, 3.875, 4.2)$
$i=2$	$(3, 3.85, 4.35, 4.8)$	$i=6$	$(0.6, 0.975, 1.425, 2.2)$
$i=3$	$(1.6, 2.375, 3.025, 3.6)$	$i=7$	$(1, 1.525, 1.875, 2.6)$
$i=4$	$(3.2, 3.9, 4.5, 5)$	$i=8$	$(1, 1.4, 1.6, 2.4)$

Therefore, A_3 provides the ranking (linear order):

$$x_4 \; P_3 \; x_2 \; P_3 \; x_5 \; P_3 \; x_3 \; P_3 \; x_1 \; P_3 \; x_7 \; P_3 \; x_8 \; P_3 \; x_6.$$

Summarizing, the three aggregation functions provide three different complete preorders (in fact linear orders). Only the fourth, seventh and eighth positions are identical in the three rankings.

4. CHOOSING A FINAL OUTCOME

Taking into account the outcomes given by a particular aggregator, we need to choose a final outcome. In order to do this, we will consider an aggregation rule which give us a single complete preorder representing the group opinion.

An aggregation rule $F : R(X)^r \longrightarrow R(X)$ is a function which assigns a complete preorder R on X to each family of r complete preorders. We suppose that this family is the outcome of an aggregator $\mathbf{A} : \nu \longrightarrow R(X)^r$. Combining both aggregation procedures we have a decision rule $F_A = F \circ \mathbf{A} : \nu \longrightarrow R(X)$ which provides a complete preorder, the final outcome, to each profile of individual linguistic assessments.

We will consider weights $w_1,\ldots,w_r > 0$, such that $w_1 + \ldots + w_r = 1$, in order to associate different importance to the aggregation functions $A_1,\ldots,A_r$. Taking into account $R_k = A_k(v_1,\ldots,v_m)$, the complete preorder given by A_k to the profile $(v_1,\ldots,v_m)$, let P_k and $\mathbf{I}_k$ be the strong preference relation and the indifference relation associated with R_k, respectively. Now we define

$$f_k(x_i) = \text{card } \{x_j \mid x_i P_k x_j\} ,$$

the Borda score given by R_k to alternative x_i.

We note that, by transitivity of P_k, $f_k(x_i) > f_k(x_j)$ whenever $x_i P_k x_j$ (see García-Lapresta and Martínez-Panero (2002)).

The total score of alternative x_i is defined by

$$f(x_i) = \sum_{k=1}^{r} w_k \cdot f_k(x_i) .$$

Thus, we obtain the final complete preorder

$$R = F(R_1,\ldots,R_r) = F_A(v_1,\ldots,v_m)$$

defined by

$$x_i R x_j \Leftrightarrow f(x_i) \geq f(x_j).$$

Example 3. We now consider the three aggregation functions included in the previous example and their respective outcomes:

1. $A_1 : x_5 \ P_1 \ x_2 \ P_1 \ x_4 \ P_1 \ x_3 \ P_1 \ x_7 \ P_1 \ x_1 \ P_1 \ x_8 \ P_1 \ x_6.$
2. $A_2 : x_2 \ P_2 \ x_5 \ P_2 \ x_4 \ P_2 \ x_3 \ P_2 \ x_7 \ P_2 \ x_1 \ P_2 \ x_8 \ P_2 \ x_6.$
3. $A_3 : x_4 \ P_3 \ x_2 \ P_3 \ x_5 \ P_3 \ x_3 \ P_3 \ x_1 \ P_3 \ x_7 \ P_3 \ x_8 \ P_3 \ x_6.$

As an example we define three aggregation rules $F_i : R(X)^r \longrightarrow R(X)$, $i = 1,2,3,$ associated with different weights. We denote $F_i(R_1,R_2,R_3) = \bar{R}_i$; $\bar{P}_i$ and $\bar{I}_i$ are the strong preference relation and the indifference relation associated with $\bar{R}_i$, respectively. Now we show the final outcome provided by each one of these aggregation rules.

1. F_1 is defined by the weights $w_1 = w_2 = w_3 = \frac{1}{3}$. The total scores of the alternatives are:

$f(x_1) = 2.\widehat{3}$	$f(x_3) = 4$	$f(x_5) = 6$	$f(x_7) = 2.\widehat{6}$
$f(x_2) = 6.\widehat{3}$	$f(x_4) = 5.\widehat{6}$	$f(x_6) = 0$	$f(x_8) = 1.$

   Therefore, the complete preorder obtained through F_1 is:

   $$x_2 \ \bar{P}_1 \ x_5 \ \bar{P}_1 \ x_4 \ \bar{P}_1 \ x_3 \ \bar{P}_1 \ x_7 \ \bar{P}_1 \ x_1 \ \bar{P}_1 \ x_8 \ \bar{P}_1 \ x_6.$$

2. F_2 is defined by the weights $w_1 = 0.6,\ w_2 = w_3 = 0.2$. The total scores of the alternatives are:

$f(x_1) = 2.2$	$f(x_3) = 4$	$f(x_5) = 6.4$	$f(x_7) = 2.8$
$f(x_2) = 6.2$	$f(x_4) = 5.4$	$f(x_6) = 0$	$f(x_8) = 1.$

   Consequently, the complete preorder obtained through F_2 is:
   $$x_5 \ \bar{P}_2 \ x_2 \ \bar{P}_2 \ x_4 \ \bar{P}_2 \ x_3 \ \bar{P}_2 \ x_7 \ \bar{P}_2 \ x_1 \ \bar{P}_2 \ x_8 \ \bar{P}_2 \ x_6.$$

3. F_3 is defined by the weights $w_1 = 0.5,\ w_2 = 0.3,\ w_3 = 0.2$. The total scores of the alternatives are:

$f(x_1) = 2.2$	$f(x_3) = 4$	$f(x_5) = 6.3$	$f(x_7) = 2.8$
$f(x_2) = 6.3$	$f(x_4) = 5.4$	$f(x_6) = 0$	$f(x_8) = 1.$

Therefore, the complete preorder obtained through F_3 is:

$$x_5 \ \bar{I}_3 \ x_2 \ \bar{P}_3 \ x_4 \ \bar{P}_3 \ x_3 \ \bar{P}_3 \ x_7 \ \bar{P}_3 \ x_1 \ \bar{P}_3 \ x_8 \ \bar{P}_3 \ x_6.$$

We note that the top of the three final complete preorders are different: the first one ranks x_2 over x_5, the second one ranks x_5 over x_2; and the third one ranks x_5 and x_2 at the same level. Clearly, the weights with which we give different importance to the aggregation functions are crucial in the final outcomes provided by the decision rules.

Acknowledgements

This work is partially financed by Junta de Castilla y León (Consejería de Educación y Cultura, Proyecto VA057/02), Ministerio de Ciencia y Tecnología, Plan Nacional de Investigación, Desarrollo e Innovación Tecnológica (I+D+I) (Proyecto BEC2001-2253) and ERDF.

REFERENCES

BLACK, D. (1976). “Partial justification of the Borda count”. *Public Choice* 28, p. 1-16.
BRAMS, S.J.; FISHBURN, P.C. (1978). “Approval voting”. *American Political Science Review* 72, p. 831-847.
BRAMS, S.J.; FISHBURN, P.C. (1983). *Approval Voting.* Boston, Birkhäuser
DELGADO, M.; VILA, M.A.; VOXMAN, W. (1998). “On a canonical representation of fuzzy numbers”. *Fuzzy Sets and Systems* 93, p. 125-135.
GARCÍA-LAPRESTA, J.L. (forthcoming). “A general class of simple majority decision rules based on linguistic opinions”. *Information Sciences.*
GARCÍA-LAPRESTA, J.L. (2003). “Linguistic assessments and rankings in two-stage group decision making procedures”. *Proceedings of the 7th Joint Conference on Information Sciences, Research Triangle Park*, North Carolina, USA, p. 143-146.
GARCÍA-LAPRESTA, J.L.; LAZZARI, L.L.; MARTÍNEZ-PANERO, M. (2001). “A group decision making method using fuzzy triangular numbers” in: Zopounidis, C.; Pardalos, P.M.; Baourakis, G. (eds.). *Fuzzy Sets in Management, Economics and Marketing.* Singapore, World Scientific, p. 35-50.
GARCÍA-LAPRESTA, J.L.; MARTÍNEZ-PANERO, M. (2002). “Borda count versus approval voting: a fuzzy approach”. *Public Choice* 112, p. 167-184.
GÄRDENFORS, P. (1973). “Positionalist voting functions”. *Theory and Decision* 4, p. 1-24.
HERRERA, F.; HERRERA-VIEDMA, E. (2000). “Linguistic decision analysis: Steps for solving decision problems under linguistic information”. *Fuzzy Sets and Systems* 115, p. 67-82.
HERRERA, F.; HERRERA-VIEDMA, E.; VERDEGAY, J.L. (1996). “Direct approach processes in group decision making using linguistic OWA operators”. *Fuzzy Sets and Systems* 79, p. 175-190.
MARIMIN; UMANO, M.; HATONO, I.; TAMURA, H. (1998). “Linguistic labels for expressing fuzzy preference relations in fuzzy group decision making”. *IEEE Transactions on Systems, Man, and Cybernetics* - Part B: Cybernetics 28, p. 205-218.
NITZAN, S.; RUBINSTEIN, A. (1981). “A further characterization of Borda ranking method”. *Public Choice* 36, p. 153-158.
YLMAZ, M.R. (1999). “Can we improve upon approval voting?” *European Journal of Political Economy* 15, p. 89-100.
ZADEH, L.A. (1975). “The concept of a linguistic variable and its applications to approximate reasoning”. *Information Sciences*. Part I: 8, p. 199-249. Part II: 8, p. 301-357. Part III: 9, p. 43-80.