Collective transitivity in majorities based on difference in support

Bonifacio Llamazares[a,*], Patrizia Pérez-Asurmendi[b], José Luis García-Lapresta[a]

[a] *PRESAD Research Group, IMUVA, Dep. of Applied Economics, University of Valladolid, Avda. Valle de Esgueva 6, 47011 Valladolid, Spain.*
[b] *PRESAD & SEED Research Groups, Dep. of Applied Economics I, University Rey Juan Carlos, Paseo Artilleros s/n, 28032 Madrid, Spain.*

Abstract

A common criticism to simple majority voting rule is the slight support that such rule demands to declare an alternative as a winner. Among the distinct majority rules used for diminishing this handicap, we focus on majorities based on difference in support. With these majorities, voters are allowed to show intensities of preference among alternatives through reciprocal preference relations. These majorities also take into account the difference in support between alternatives in order to select the winner. In this paper we have provided some necessary and sufficient conditions for ensuring transitive collective decisions generated by majorities based on difference in support for all the profiles of individual reciprocal preference relations. These conditions involve both the thresholds of support and some individual rationality assumptions that are related to transitivity in the framework of reciprocal preference relations.

Keywords: Reciprocal preference relations, Voting systems, Majorities based on difference in support, Transitivity.

1. Introduction

Majority voting systems are doubtless the most popular methods, within some organizations, of aggregating individual opinions in order to make collective decisions Nowadays, in almost all democratic parliaments and institutions, decisions over collective issues are usually taken through majority systems as simple majority, absolute majority or qualified majorities. In fact, some authors argue that majority rules represent voters' views better than other voting systems (see Dasgupta and Maskin [7]).

From a practical point of view, these methods are easy to understand by the voters: Given two alternatives, each individual casts a vote for his/her preferred alternative or he/she abstains when he/she is indifferent between them (if it is allowed). The social decision simply consists of the selection of the most preferred alternative of a predetermined majority of the voters, if such an alternative exists.

*Corresponding author
Email addresses: `boni@eco.uva.es` (Bonifacio Llamazares), `patrizia.perez@urjc.es` (Patrizia Pérez-Asurmendi), `lapresta@eco.uva.es` (José Luis García-Lapresta)

These facts have promoted a broad literature about all those kinds of majority systems; their properties, advantages and drawbacks have been deeply studied in almost all of them. Moreover, majority rules are usually set against rank order voting systems. Only majority systems lead to decisions independent of irrelevant alternatives; however, rank order voting systems fulfil collective transitivity whereas majority systems do not. In other words, cycles on collective decision are possible under majority rules. It is commonly known that these cycles represent one of the most troublesome problems on Voting Theory given that their occurrence leads to the impossibility of getting a social outcome. The well-known voting paradox (Condorcet [4]) describes the pointed problem; whenever a social choice involves three or more alternatives and three or more voters, cycles might appear.

A common feature of majority rules and other classic voting systems, is that they require individuals to declare dichotomous preferences. In other words, voters can only declare if an alternative is preferred to another, or if they are indifferent. All kinds of preference modalities are identified and voters' opinions are misrepresented. In this way, some authors have pointed out the necessity of having more information about individuals preferences.

Quoting the Nobel Prize Laureate Amartya K. Sen [44, p. 162]: *... the method of majority decision takes no account of intensities of preference, and it is certainly arguable that what matters is not merely the <u>number</u> who prefer x to y and the <u>number</u> who prefer y to x, but also by how much each prefers one alternative to the other.* This idea had already been considered in the 18th Century by the Spanish mathematician Morales [40] who stated that *opinion is not something that can be quantified but rather something which has to be weighed* (see English translation in McLean and Urken [39, p. 204], or *... majority opinion ... is something which is independent of any fixed number of votes* (see English translation in McLean and Urken [39, p. 214]).

The importance of considering intensities of preference in the design of appropriate voting systems has also been advocated by Nurmi [42]. Following this approach, García-Lapresta and Llamazares [23] provided some axiomatic characterizations of several decision rules that aggregate reciprocal preferences through different kind of means. Moreover, in García-Lapresta and Llamazares [23, Prop. 2], simple majority has been obtained as a specific case of some means-based decision rules. Likewise, other kinds of majorities can be obtained through operators that aggregate reciprocal preferences (on this, see Llamazares and García-Lapresta [36, 37] and Llamazares [33, 35]).

Majority rules based on difference of votes were introduced by García-Lapresta and Llamazares [24] in order to avoid some of the drawbacks of simple and absolute majorities. Furthermore, in García-Lapresta and Llamazares [25] these majority rules were extended by allowing individuals to show their intensities of preference among alternatives. They introduced *majorities based on difference in support* or $\widetilde{M}_k$ *majorities* and provided a characterization by means of some independent axioms.

As previously mentioned, the voting paradox constitutes a key aspect of voting systems, and it is also crucial in this research. We devote this paper to analyze when majorities based on difference in support ($\widetilde{M}_k$

majorities) provide transitive collective preference relations for every profile of individual reciprocal preferences satisfying some rationality conditions. In other words, we study when the aggregation of individual intensities of preferences through $\widetilde{M}_k$ majorities provides transitive social decisions.

The paper is organized as follows. Section 2 is devoted to introducing some classes of transitivity conditions in the field of reciprocal preferences. Moreover, some well-known majority rules are reviewed, and the extension of majorities based on difference of votes to the field of reciprocal preferences, the class of majorities based on difference in support, is introduced. Section 3 includes the results and some illustrative examples. Finally, Section 4 provides some concluding remarks.

2. Preliminaries

Consider m voters, $V = \{1, \dots, m\}$, with $m \geq 2$, showing the intensity of their preferences on n alternatives, $X = \{x_1, \dots, x_n\}$, with $n \geq 2$, through *reciprocal* preference relations $R^p : X \times X \longrightarrow [0, 1]$, for $p = 1, \dots, m$, i.e., $r^p_{ij} + r^p_{ji} = 1$ for all $i, j \in \{1, \dots, n\}$, where $r^p_{ij} = R^p(x_i, x_j)$. The information contained in R^p can be represented by a $n \times n$ matrix with coefficients in $[0, 1]$

$$R^p = \begin{pmatrix} r^p_{11} & r^p_{12} & \cdots & r^p_{1n} \\ r^p_{21} & r^p_{22} & \cdots & r^p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r^p_{n1} & r^p_{n2} & \cdots & r^p_{nn} \end{pmatrix}$$

where, by reciprocity, all the diagonal elements are 0.5 and $r^p_{ji} = 1 - r^p_{ij}$ if $j \neq i$. Rewriting the matrix according to these facts, we have:

$$R^p = \begin{pmatrix} 0.5 & r^p_{12} & \cdots & r^p_{1n} \\ 1 - r^p_{12} & 0.5 & \cdots & r^p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 1 - r^p_{1n} & 1 - r^p_{2n} & \cdots & 0.5 \end{pmatrix}.$$

Under this setting, we assume that voters are able to distinguish whether they prefer one alternative to another or if they are totally indifferent between them. We also consider that voters may provide numerical degrees of preference among the alternatives by means of numbers within a bipolar scale in the unit interval. More specifically, given two alternatives x_i and x_j, if voter p is indifferent between these alternatives, then $r^p_{ij} = 0.5$. But if this individual prefers an alternative to the other, then he/she can show the intensity of preference in the following way: $r^p_{ij} = 0$, when p absolutely prefers x_j to x_i; $r^p_{ij} = 1$, when p absolutely prefers x_i to x_j. If p does not declare extreme preferences or indifference, then r^p_{ij} takes some value between 0 and 1 different to 0.5. If this voter somewhat prefers x_i to x_j, then $0.5 < r^p_{ij} < 1$, and the closer this number is to 1, the more x_i is preferred to x_j. A similar interpretation can be done for values located between 0 and 0.5: if p somewhat prefers x_j to x_i, then $0 < r^p_{ij} < 0.5$, and the closer this number is to 0, the more x_j is preferred to x_i (see Nurmi [41] and García-Lapresta and Llamazares [23]).

Notice that reciprocity extends two properties from the framework of ordinary preferences to the context of intensities of preference: asymmetry (see García-Lapresta and Meneses [27]) and completeness (see De Baets and De Meyer [8]).

With $\mathcal{R}(X)$ we denote the set of reciprocal preference relations on X. A *profile* is a vector $(R^1, \ldots, R^m)$ containing the individual reciprocal preference relations on X. Accordingly, the set of profiles is denoted by $\mathcal{R}(X)^m$.

An *ordinary preference relation* on X is an *asymmetric* binary relation on X: if $x_i \, P \, x_j$, then does not happen $x_j \, P \, x_i$. The indifference relation associated with P is defined as $x_i \, I \, x_j$ if neither $x_i \, P \, x_j$ nor $x_j \, P \, x_i$. With $\mathcal{P}(X)$ we denote the set of ordinary preference relations on X.

Notice that every ordinary preference relation P can be considered as a reciprocal preference relation R:

$$\begin{aligned} x_i \, P \, x_j \quad &\Leftrightarrow \quad r_{ij} = 1 \\ x_j \, P \, x_i \quad &\Leftrightarrow \quad r_{ij} = 0 \\ x_i \, I \, x_j \quad &\Leftrightarrow \quad r_{ij} = 0.5\,. \end{aligned}$$

On the other hand, a reciprocal preference relation R^p is *crisp* if $r^p_{ij} \in \{0, 0.5, 1\}$ for all $i, j \in \{1, \ldots, n\}$. So, in practice, crisp reciprocal preference relations and ordinary preference relations are equivalent.

An ordinary preference relation $P \in \mathcal{P}(X)$ is *transitive* if for all $x_i, x_j, x_l \in X$ it holds that if $x_i \, P \, x_j$ and $x_j \, P \, x_l$, then it also holds $x_i \, P \, x_l$.

In Figure 1, the difference on the information reported by ordinary preferences in comparison with reciprocal preferences is shown.

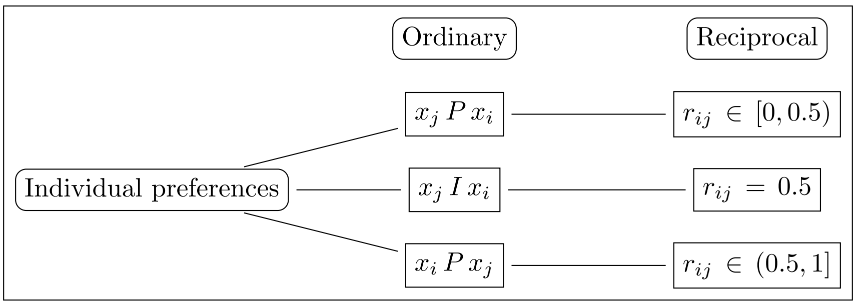

Figure 1: Ordinary *versus* reciprocal preferences.

2.1. Consistency on reciprocal preference relations

Within the framework of ordinary preference relations, the most well-known rationality assumption is transitivity. However, such a condition could be extended in many different ways when considering reciprocal preference relations or similar structures (see, for instance, Zadeh [51], Dubois and Prade [17], Tanino [49], Jain [31], De Baets *et al.* [12], De Baets and Van de Walle [11], Dasgupta and Deb [6], Van de Walle *et al.* [50], Switalski [46, 47], Herrera-Viedma *et al.* [29], Díaz *et al.* [15, 16, 13, 14], De Baets and De Meyer [8], García-Lapresta and Meneses [26], De Baets *et al.* [10], García-Lapresta and Montero [28], Chiclana *et al.* [2, 3], Alonso *et al.* [1] or De Baets *et al.* [9]).

It is important to note that in 1973 Fishburn [20] provided some extensions of transitivity within the probabilistic choice framework. These extensions can be considered as precursors of some transitivity properties in the field of reciprocal relations.

We now introduce the notion of transitivity we use in the results (see also Figure 2).

Definition 1. A function $g : [0.5, 1]^2 \longrightarrow [0.5, 1]$ is a *monotonic operator* if it is satisfies the following conditions:

1. Continuity.
2. Increasingness: $g(a, b) \geq g(c, d)$ for all $a, b, c, d \in [0.5, 1]$ such that $a \geq c$ and $b \geq d$.
3. Symmetry: $g(a, b) = g(b, a)$ for all $a, b \in [0.5, 1]$.

Definition 2. Given a monotonic operator g, $R \in \mathcal{R}(X)$ is *g-transitive* if for all $i, j, l \in \{1, \ldots, n\}$ the following holds:

$$\big(r_{ij} > 0.5 \text{ and } r_{jl} > 0.5\big) \Rightarrow \big(r_{il} > 0.5 \text{ and } r_{il} \geq g(r_{ij}, r_{jl})\big).$$

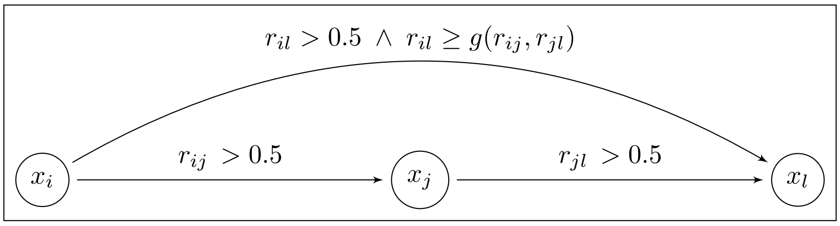

Figure 2: The notion of g-transitivity.

With T_g we denote the set of all g-transitive reciprocal preference relations. Notice that if f and g are two monotonic operators such that $f \leq g$, i.e., $f(a, b) \leq g(a, b)$ for all $a, b \in [0.5, 1]$, then $T_g \subseteq T_f$.

Figure 2 illustrates g-transitivity. This definition of transitivity allows us to distinguish between different degrees of individual's rationality. Furthermore, an appropriate choice of g allows us avoid situations where individuals' rationality could be questioned. For instance, consider the values $r^p_{ij} = 1$, $r^p_{jl} = 1$ and $r^p_{il} = 0.51$; that is, voter p absolutely prefers x_i to x_j and x_j to x_l, but only slightly x_i to x_l. In this case, the rationality of individual p could be questioned given that, in the ordinary case, the notion of transitivity means that if $x_i \, P \, x_j$ $(r^p_{ij} = 1)$ and $x_j \, P \, x_l$ $(r^p_{jl} = 1)$, then $x_i \, P \, x_l$ $(r^p_{il} = 1)$.

We now consider three of the most commonly used transitivity conditions on reciprocal preference relations by means of the monotonic operators minimum, arithmetic mean and maximum:

1. R is *min-transitive* if R is g-transitive being $g(a, b) = \min\{a, b\}$ for all $(a, b) \in [0.5, 1]^2$.
2. R is *am-transitive* if R is g-transitive being $g(a, b) = (a + b)/2$ for all $(a, b) \in [0.5, 1]^2$.
3. R is *max-transitive* if R is g-transitive being $g(a, b) = \max\{a, b\}$ for all $(a, b) \in [0.5, 1]^2$.

We denote with $T_{\min}$, T_{am} and $T_{\max}$ the sets of all min-transitive, am-transitive and max-transitive reciprocal preference relations, respectively. Clearly, $T_{\max} \subset T_{\text{am}} \subset T_{\min}$.

2.2. Majority rules

Among the wide variety of majority rules, *simple majority* holds a primary position. Recalling the rule, one alternative, say x, defeats another, say y, when the number of individuals who prefer x to y is greater than the number of individuals who prefer y to x. Since it was first characterized by May [38], a wide research has been done with regard to analyzing its properties. In particular, simple majority is the most decisive majority rule when the alternatives are equally treated. On the one hand, this constitutes an advantage with respect to other rules. On the other hand, unfortunately, it also represents an important drawback: simple majority requires a really poor support for declaring an alternative as the winner. For instance, when all voters but one abstain, the winner is the alternative receiving that single vote.

In an attempt to get a better performance on that issue, other majorities have been introduced and studied in the literature. Among them one can find unanimous majority, qualified majorities and absolute majority (see Fishburn [19, chapter 6], Ferejohn and Grether [18], Saari [43, pp. 122–123], and García-Lapresta and Llamazares [24], among others).

According to *unanimous majority*, an alternative, say x, defeats another one, say y, when every voter involved in the election casts his/her ballot for the alternative x. Obviously reaching a winner is very difficult in practice under this rule; if there is just one discordant voter, a collective choice becomes impossible.

Qualified majorities require support for the winner alternative to be greater than or equal to a quota, fixed before the election, multiplied by the number of voters. So, an alternative x defeats another alternative y by a qualified majority of quota $\alpha > 0.5$, when at least $100\,\alpha\,\%$ of voters prefer x to y. Common examples are three-fifths, two-thirds or three-quarters majorities. Note that when the required quota is 1, we are going back to unanimous majority.

Finally, when the required support for the winning alternative is greater than half the number of voters, we are looking at *absolute majority.* Obviously whenever every individual involved in the election has strict preferences over alternatives (that is, no individual is indifferent between distinct alternatives), simple majority and absolute majority are equivalent.

In García-Lapresta and Llamazares [24], another class of majorities have been introduced and analyzed: *majorities based on difference of votes* or M_k *majorities.* According to these majorities, given two alternatives, x and y, x is collectively preferred to y, when the number of individuals who prefer x to y exceeds the number of individuals who prefer y to x by at least a fixed integer k from 0 to $m-1$. These majorities have been axiomatically characterized by Llamazares [34] and subsequently by Houy [30].

We note that M_k majorities are located between simple majority and unanimity. In particular, we face simple majority when the required threshold k equals 0 whereas unanimity is reached when k equals $m-1$. Therefore the above mentioned problem of support in the case of simple majority can be solved with these rules by using appropriate thresholds. Since voters are sometimes indifferent between some alternatives, M_k majorities have an important role in group decision making.

Qualified majorities, however, are located between absolute majority and unanimity. It is interesting to note that M_k majorities and qualified majorities become equivalent when indifference is ruled out from individual preferences.

In Figure 3 all these facts are summarized, and the difference between M_k-majorities and qualified majorities, when voters can declare indifference between alternatives, is emphasized.

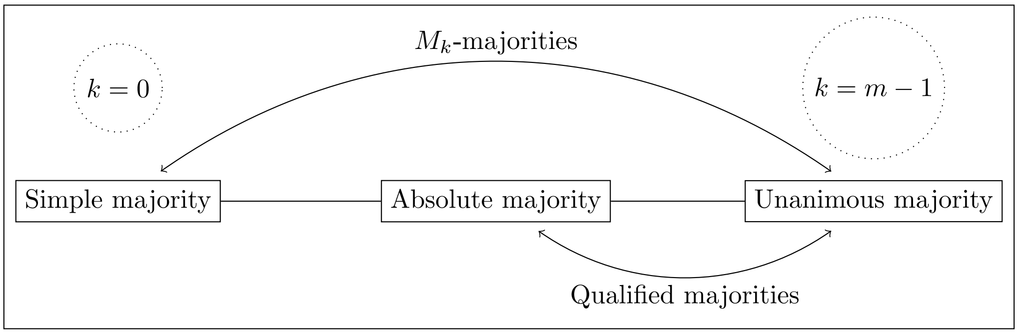

Figure 3: M_k-majorities *versus* qualified majorities.

2.3. Majorities based on difference in support

To finish this preliminary section, we now present the voting rules becoming goal of this research: majorities based on difference in support. These majorities extend the family of majorities based on difference of votes to the field of reciprocal preferences. Given two alternatives x and y, x defeats y when the aggregated intensity of preference of x over y exceeds the aggregated intensity of y over x in a threshold k fixed before the election, where k is a real number located between 0 and the total number of voters.

Definition 3. Given a threshold $k \in [0,m)$ and $\mathcal{D} \subseteq \mathcal{R}(X)^m$, the $\widetilde{M}_k$ *majority* is the mapping $\widetilde{M}_k : \mathcal{D} \longrightarrow \mathcal{P}(X)$ defined by $\widetilde{M}_k(R^1,\ldots,R^m) = P_k$, where

$$x_i \, P_k \, x_j \;\Leftrightarrow\; \sum_{p=1}^{m} r_{ij}^p > \sum_{p=1}^{m} r_{ji}^p + k.$$

Notice that $\widetilde{M}_k$ assigns a collective ordinary preference relation to each profile of individual reciprocal preference relations. It is easy to see (García-Lapresta and Llamazares [25]) that P_k can be defined through the average of the individual intensities of preference:

$$x_i \, P_k \, x_j \;\Leftrightarrow\; \frac{1}{m}\sum_{p=1}^{m} r_{ij}^p > \frac{m+k}{2m},$$

or, equivalently,

$$x_i \, P_k \, x_j \;\Leftrightarrow\; \sum_{p=1}^{m} r_{ij}^p > \frac{m+k}{2}. \tag{1}$$

We can rewrite P_k by means of an α–cut[1]. Let $\overline{R} : X \times X \longrightarrow [0,1]$ the reciprocal preference relation defined by the arithmetic mean of the individual intensities of preference, i.e.,

$$\overline{R}(x_i,x_j) = \frac{1}{m}\sum_{p=1}^{m} r_{ij}^p.$$

Then, $P_k = \overline{R}_\alpha$, with $\alpha = (m+k)/2m$.

The indifference relation associated with P_k is defined by:

$$x_i \, I_k \, x_j \;\Leftrightarrow\; \left| \sum_{p=1}^{m} r_{ij}^p - \sum_{p=1}^{m} r_{ji}^p \right| \leq k,$$

or, equivalently,

$$x_i \, I_k \, x_j \;\Leftrightarrow\; \left| \sum_{p=1}^{m} r_{ij}^p - \frac{m}{2} \right| \leq \frac{k}{2}. \tag{2}$$

Going deeper on the behavior of $\widetilde{M}_k$ majorities, some interesting facts can be detailed.

Remark 1. For $\widetilde{M}_k$ majorities, the following statements hold:

1. $(\, x_i \, P_k \, x_j$ and $k' > k) \;\Rightarrow\; (\, x_i \, P_{k'} \, x_j$ or $x_i \, I_{k'} \, x_j \,)$.
2. $(\, x_i \, P_k \, x_j$ and $k' < k \,) \;\Rightarrow\; x_i \, P_{k'} \, x_j$.
3. $(\, \exists\, k \in [0,m)\; x_i \, P_k \, x_j \,) \;\Rightarrow\; (\forall\, k' \in [0,m)\; \neg(x_j \, P_{k'} \, x_i))$.

The first implication states that whenever an alternative is preferred to another one for a given threshold, that preference holds or, at most, turns into an indifference for any other threshold greater than the first considered one. The second fact shows that the preference between two alternatives does not change when the threshold becomes smaller. The third statement is a consequence of the previous two: Whenever one alternative is preferred to another for a certain threshold, that preference cannot be reversed for neither a greater nor a smaller threshold.

[1] If $R \in \mathcal{R}(X)$ and $\alpha \in [0.5,1)$, the α–cut of R is the ordinary preference relation R_α defined by $x_i \, R_\alpha \, x_j \;\Leftrightarrow\; R(x_i,x_j) > \alpha$.

3. Ensuring transitive collective decisions

This section includes the results and some illustrative examples. We establish necessary and sufficient conditions on thresholds k for ensuring that majorities based on difference in support provide transitive collective preferences P_k for every profile of several types of individual reciprocal preference relations. To be more specific, we assume different kinds of rationality assumptions to represent the voter's behavior.

In the following proposition we establish a necessary condition on thresholds k in $\widetilde{M}_k$ majorities for having transitive collective preference relations for every profile of reciprocal preference relations satisfying any kind of g-transitivity. This necessary condition is very restrictive: k should be at least $m-1$.

Proposition 1. *If g is a monotonic operator, then there does not exist $k \in [0, m-1)$ such that P_k is transitive for every profile of individual preferences $(R^1, \dots, R^m) \in T_g^m$.*

Proof. Let g be a monotonic operator, $k \in [0, m-1)$ and R', R'' be the following reciprocal preference relations:

$$R' = \begin{pmatrix} 0.5 & 1 & 1 & \dots \\ 0 & 0.5 & 1 & \dots \\ 0 & 0 & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \qquad R'' = \begin{pmatrix} 0.5 & 0.5 & 0 & \dots \\ 0.5 & 0.5 & 0.5 & \dots \\ 1 & 0.5 & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

where non clearly stated elements above take the value 0.5. It is not difficult to prove that $R', R'' \in T_g$.

Let consider the preference profile $(R^1, \dots, R^m)$, where[2]

$$R^i = \begin{cases} R', & \text{if } i = 1, \dots, \left\lfloor \frac{m+k}{2} \right\rfloor, \\ R'', & \text{if } i = \left\lfloor \frac{m+k}{2} \right\rfloor + 1, \dots, m. \end{cases}$$

According to expression (1), $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$ will happen if the following is true:

$$\left\lfloor \frac{m+k}{2} \right\rfloor + \frac{1}{2} \left(m - \left\lfloor \frac{m+k}{2} \right\rfloor \right) > \frac{m+k}{2},$$

or, equivalently, $\left\lfloor \frac{m+k}{2} \right\rfloor > k$. Let see that such condition is always fulfilled. Since $k \in [0, m-1)$, we distinguish two different cases:

1. If $k \in [m-2, m-1)$, then $\left\lfloor \frac{m+k}{2} \right\rfloor = m-1 > k$.
2. If $k < m-2$, then $2k < m+k-2$; that is, $k < \frac{m+k}{2} - 1$. Therefore, $k < \left\lfloor \frac{m+k}{2} \right\rfloor$.

Consequently, $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$. If P_k was transitive, then $x_1 \, P_k \, x_3$; that is,

$$\sum_{p=1}^{m} r_{13}^p = \left\lfloor \frac{m+k}{2} \right\rfloor > \frac{m+k}{2},$$

which is impossible. $\square$

[2] Given $x \in \mathbb{R}$, $\lfloor x \rfloor$ means the integer part of x; that is, the highest integer lower than or equal to x.

Proposition 1 establishes that is not possible to guarantee the transitivity of the collective preference relation for every profile of reciprocal preference relations whenever thresholds are smaller than $m-1$. In spite of that result, transitivity can be satisfied in some profiles as it is shown in Example 1.

Example 1. Consider the profile of three reciprocal preference relations on $X = \{x_1, x_2, x_3\}$ given by the following matrices

$$R^1 = \begin{pmatrix} 0.5 & 1 & 0.9 \\ 0 & 0.5 & 1 \\ 0.1 & 0 & 0.5 \end{pmatrix}, \qquad R^2 = \begin{pmatrix} 0.5 & 1 & 0.8 \\ 0 & 0.5 & 0.9 \\ 0.2 & 0.1 & 0.5 \end{pmatrix}, \qquad R^3 = \begin{pmatrix} 0.5 & 0.9 & 0.8 \\ 0.1 & 0.5 & 1 \\ 0.2 & 0 & 0.5 \end{pmatrix}.$$

It is easy to see that $R^1, R^2, R^3 \notin T_{\min}$.

Taking into account the definition of the collective ordinary preference relation for $\widetilde{M}_k$-majorities given by expression (1),

$$x_i \, P_k \, x_j \;\Leftrightarrow\; r^1_{ij} + r^2_{ij} + r^3_{ij} > \frac{3+k}{2},$$

we have that

$$x_1 \, P_k \, x_2 \;\Leftrightarrow\; k < 2.8, \qquad x_2 \, P_k \, x_3 \;\Leftrightarrow\; k < 2.8, \qquad x_1 \, P_k \, x_3 \;\Leftrightarrow\; k < 2\,.$$

Table 1 contains the information about the collective preferences and indifferences (see expression (2)) for all the thresholds $k \in [0, 3)$. It can be checked that P_k is transitive for every $k \in [0, 2) \cup [2.8, 3)$.

Table 1: Collective preferences in Example 1.

	x_1 *vs.* x_2	x_2 *vs.* x_3	x_1 *vs.* x_3
$0 \leq k < 2$	$x_1 \, P_k \, x_2$	$x_2 \, P_k \, x_3$	$x_1 \, P_k \, x_3$
$2 \leq k < 2.8$	$x_1 \, P_k \, x_2$	$x_2 \, P_k \, x_3$	$x_1 \, I_k \, x_3$
$2.8 \leq k < 3$	$x_1 \, I_k \, x_2$	$x_2 \, I_k \, x_3$	$x_1 \, I_k \, x_3$

In the following proposition we establish a sufficient condition on the individual rational behavior for ensuring transitive collective preference relations for every profile of individual reciprocal preferences if thresholds are at least $m-1$. Specifically, this sufficient condition requires that each individual is g-transitive, g being a monotonic operator whose values are not smaller than those given by the arithmetic mean.

Proposition 2. *For each monotonic operator g such that $g(a,b) \geq (a+b)/2$ for all $(a,b) \in [0.5, 1]^2$ and each $k \in [m-1, m)$, P_k is transitive for every profile of individual preferences $(R^1, \ldots, R^m) \in T_g^m$.*

Proof. Let g be a monotonic operator such that $g(a,b) \geq (a+b)/2$ for all $(a,b) \in [0.5,1]^2$, $k \in [m-1,m)$, $(R^1,\ldots,R^m) \in T_g^m$ and $x_i, x_j, x_l \in X$ such that $x_i\, P_k\, x_j$ and $x_j\, P_k\, x_l$. Since $k \geq m-1$, it happens that

$$\sum_{p=1}^{m} r_{ij}^p > \frac{m+k}{2} \geq m - \frac{1}{2} \quad \text{and} \quad \sum_{p=1}^{m} r_{jl}^p > \frac{m+k}{2} \geq m - \frac{1}{2}.$$

For inequalities above to be true, it is necessary that every addend be greater than 0.5; that is, $r_{ij}^p > 0.5$ and $r_{jl}^p > 0.5$ for all $p \in \{1,\ldots,m\}$.

Since $(R^1,\ldots,R^m) \in T_g^m$, it happens that $r_{il}^p \geq g(r_{ij}^p, r_{jl}^p) \geq (r_{ij}^p + r_{jl}^p)/2$ for all $p \in \{1,\ldots,m\}$. Therefore,

$$\sum_{p=1}^{m} r_{il}^p \geq \sum_{p=1}^{m} \frac{r_{ij}^p + r_{jl}^p}{2} = \frac{1}{2}\left(\sum_{p=1}^{m} r_{ij}^p + \sum_{p=1}^{m} r_{jl}^p\right) > \frac{1}{2}\left(\frac{m+k}{2} + \frac{m+k}{2}\right) = \frac{m+k}{2};$$

that is, $x_i\, P_k\, x_l$. $\square$

Proposition 2 ensures the transitivity of the collective preference relation for every profile of reciprocal preference relations satisfying some transitivity conditions, whenever thresholds are at least $m-1$. Example 2 shows how inconsistencies diminish when thresholds increase.

Example 2. Consider the profile of three reciprocal preference relations on $X = \{x_1, x_2, x_3\}$ given by the following matrices

$$R^1 = \begin{pmatrix} 0.5 & 1 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0.5 \end{pmatrix}, \qquad R^2 = \begin{pmatrix} 0.5 & 0.1 & 0.1 \\ 0.9 & 0.5 & 0.9 \\ 0.9 & 0.1 & 0.5 \end{pmatrix}, \qquad R^3 = \begin{pmatrix} 0.5 & 0.9 & 0.2 \\ 0.1 & 0.5 & 0.1 \\ 0.8 & 0.9 & 0.5 \end{pmatrix}.$$

It can be tested that $R^1, R^2, R^3 \in T_{\max}$.

Taking into account expression (1),

$$x_i\, P_k\, x_j \;\Leftrightarrow\; r_{ij}^1 + r_{ij}^2 + r_{ij}^3 > \frac{3+k}{2},$$

we have that

$$x_1\, P_k\, x_2 \;\Leftrightarrow\; k < 1, \qquad x_2\, P_k\, x_3 \;\Leftrightarrow\; k < 1, \qquad x_3\, P_k\, x_1 \;\Leftrightarrow\; k < 0.4.$$

In Table 2 we present the information about the collective preferences and indifferences (see expression (2)) for all the thresholds $k \in [0,3)$. In this case, P_k is not transitive for every $k \in [0,1)$. Specifically, if $k < 0.4$, we have a cycle, whereas if $k \in [0.4,1)$ the cycle disappears but transitivity is not satisfied. If $k \geq 1$, all three alternatives are declared socially indifferent, so transitivity is trivially fulfilled.

In the following proposition we establish that the sufficient condition given in Proposition 2 is also necessary. Furthermore, if it is not fulfilled, then it is not possible to guarantee collective transitivity for every profile and any threshold.

Table 2: Collective preferences in Example 2.

	x_1 *vs.* x_2	x_2 *vs.* x_3	x_1 *vs.* x_3
$0 \leq k < 0.4$	$x_1 \, P_k \, x_2$	$x_2 \, P_k \, x_3$	$x_3 \, P_k \, x_1$
$0.4 \leq k < 1$	$x_1 \, P_k \, x_2$	$x_2 \, P_k \, x_3$	$x_1 \, I_k \, x_3$
$1 \leq k < 3$	$x_1 \, I_k \, x_2$	$x_2 \, I_k \, x_3$	$x_1 \, I_k \, x_3$

Proposition 3. *For each monotonic operator g such that $g(a,b) < (a+b)/2$ for all $(a,b) \in [0.5,1]^2$ with $a \neq b$, there does not exist $k \in [m-1,m)$ such that P_k is transitive for every profile of individual preferences $(R^1,\dots,R^m) \in T_g^m$.*

Proof. Let g be a monotonic operator such that $g(a,b) < (a+b)/2$ for all $(a,b) \in [0.5,1]^2$ with $a \neq b$, and let $h : [0.5,1] \longrightarrow [0.5,1]$ be such that $h(a) = g(a,1)$ for all $a \in [0.5,1]$. By construction, h is continuous and $h(0.5) = g(0.5,1) < (0.5+1)/2 = 0.75$. Moreover, given $k \in [m-1,m)$, we have that $(m+k)/2m \geq (2m-1)/2m$. Since $m \geq 2$, $(2m-1)/2m \geq 0.75$. Therefore, $(m+k)/2m \geq 0.75$ and $h(0.5) < (m+k)/2m$.

We distinguish two cases:

1. If $h(1) \leq (m+k)/2m$, we consider the profile of individual preferences $(R^1,\dots,R^m)$ such that, for all $i \in \{1,\dots,m\}$,

$$R^i = \begin{pmatrix} 0.5 & 1 & \frac{m+k}{2m} & \dots \\ 0 & 0.5 & 1 & \dots \\ \frac{m-k}{2m} & 0 & 0.5 & \dots \\ \multicolumn{4}{c}{\dotfill} \end{pmatrix}.$$

   Again, as in the proof of Proposition 1, the non clearly stated elements take the value 0.5. It is easy to check that $(R^1,\dots,R^m) \in T_g^m$. By expression (1), it is clear that $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$. Since

$$\sum_{p=1}^{m} r_{13}^p = m\frac{m+k}{2m} = \frac{m+k}{2},$$

   we have that $\neg(x_1 \, P_k \, x_3)$, that is, P_k is not transitive.

2. If $h(1) > (m+k)/2m$, since h is a continuous function and $h(0.5) < (m+k)/2m$, there exists an $a_k \in (0.5,1)$ such that $h(a_k) = g(a_k,1) = (m+k)/2m$.

Let R', R'' and R''' the reciprocal preference relations showed below:

$$R' = \begin{pmatrix} 0.5 & a_k & h(a_k) & \dots \\ 1-a_k & 0.5 & 1 & \dots \\ 1-h(a_k) & 0 & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad R'' = \begin{pmatrix} 0.5 & 1 & h(a_k) & \dots \\ 0 & 0.5 & a_k & \dots \\ 1-h(a_k) & 1-a_k & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$R''' = \begin{pmatrix} 0.5 & h(a_k) & h(a_k) & \dots \\ 1-h(a_k) & 0.5 & h(a_k) & \dots \\ 1-h(a_k) & 1-h(a_k) & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

where non clearly stated elements above take the value 0.5. It is obvious that $R', R'' \in T_g$. Moreover, since g is continuous and $g(a,b) < (a+b)/2$ for all $(a,b) \in [0.5,1]^2$ with $a \neq b$, we have that

$$g(h(a_k), h(a_k)) \leq \frac{h(a_k)+h(a_k)}{2} = h(a_k).$$

Therefore $R''' \in T_g$. We distinguish two cases:

(a) If m is even, we consider the profile of preferences $(R^1, \dots, R^m)$, where

$$R^i = \begin{cases} R', & \text{if } i = 1, \dots, \frac{m}{2}, \\ R'', & \text{if } i = \frac{m}{2}+1, \dots, m. \end{cases}$$

According to expression (1), to $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$ to be true it is necessary that

$$\frac{m}{2}a_k + \frac{m}{2} > \frac{m+k}{2}.$$

Let see that such condition is always fulfilled:

$$\frac{m}{2}a_k + \frac{m}{2} = \frac{m}{2}(a_k+1) > \frac{m}{2}(2g(a_k,1)) = m\, h(a_k) = \frac{m+k}{2}.$$

If P_k was transitive, then $x_1 \, P_k \, x_3$; in other words,

$$\sum_{p=1}^{m} r_{13}^p = m\, h(a_k) = \frac{m+k}{2} > \frac{m+k}{2},$$

which is impossible.

(b) If m is odd, we consider the profile of preferences $(R^1, \dots, R^m)$, where

$$R^i = \begin{cases} R', & \text{if } i = 1, \dots, \frac{m-1}{2}, \\ R'', & \text{if } i = \frac{m+1}{2}, \dots, m-1, \\ R''', & \text{if } i = m. \end{cases}$$

Having in mind expression (1), $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$ happen if the following is fulfilled:

$$\frac{m-1}{2} a_k + \frac{m-1}{2} + h(a_k) > \frac{m+k}{2}.$$

Let see such condition is always satisfied:

$$\begin{aligned}\frac{m-1}{2} a_k + \frac{m-1}{2} + h(a_k) &= \frac{m-1}{2}(a_k+1) + h(a_k) > \frac{m-1}{2}(2g(a_k,1)) + h(a_k) \\ &= (m-1)\, h(a_k) + h(a_k) = \frac{m+k}{2}.\end{aligned}$$

If P_k was transitive, then $x_1 \, P_k \, x_3$; that is,

$$\sum_{p=1}^{m} r_{13}^p = m\, h(a_k) = \frac{m+k}{2} > \frac{m+k}{2},$$

which is impossible. $\square$

Proposition 3 establishes that the transitivity of the collective preference relation cannot be guaranteed for every profile of reciprocal preference relations satisfying some low transitivity conditions, even when thresholds are at least $m-1$. As in Proposition 1, this result does not rule out the fact that transitivity can be satisfied in some profiles. To illustrate this, we provide the following example.

Example 3. Consider the profile of three reciprocal preference relations on $X = \{x_1, x_2, x_3\}$ given by the following matrices

$$R^1 = \begin{pmatrix} 0.5 & 1 & 0.9 \\ 0 & 0.5 & 0.9 \\ 0.1 & 0.1 & 0.5 \end{pmatrix}, \qquad R^2 = \begin{pmatrix} 0.5 & 0.9 & 0.9 \\ 0.1 & 0.5 & 1 \\ 0.1 & 0 & 0.5 \end{pmatrix}, \qquad R^3 = \begin{pmatrix} 0.5 & 1 & 0.9 \\ 0 & 0.5 & 1 \\ 0.1 & 0 & 0.5 \end{pmatrix}.$$

It can be checked that $R^1, R^2, R^3 \notin T_{\text{am}}$.

By expression (1),

$$x_i \, P_k \, x_j \;\Leftrightarrow\; r_{ij}^1 + r_{ij}^2 + r_{ij}^3 > \frac{3+k}{2},$$

we have that

$$x_1 \, P_k \, x_2 \;\Leftrightarrow\; k < 2.8, \qquad x_2 \, P_k \, x_3 \;\Leftrightarrow\; k < 2.8, \qquad x_1 \, P_k \, x_3 \;\Leftrightarrow\; k < 2.4.$$

Table 3 contains the information about the collective preferences and indifferences (see expression (2)) for all the thresholds $k \in [0,3)$. According to this situation, P_k is transitive for every $k \in [0, 2.4) \cup [2.8, 3)$.

4. Concluding remarks

In this paper we have presented some necessary and sufficient conditions under which $\widetilde{M}_k$ majorities lead to transitive collective preference relations. These results highlight the importance of the arithmetic mean for aggregating individual intensities of preference.

Table 3: Collective preferences in Example 3.

	x_1 *vs.* x_2	x_2 *vs.* x_3	x_1 *vs.* x_3
$0 \leq k < 2.4$	$x_1 \, P_k \, x_2$	$x_2 \, P_k \, x_3$	$x_1 \, P_k \, x_3$
$2.4 \leq k < 2.8$	$x_1 \, P_k \, x_2$	$x_2 \, P_k \, x_3$	$x_1 \, I_k \, x_3$
$2.8 \leq k < 3$	$x_1 \, I_k \, x_2$	$x_2 \, I_k \, x_3$	$x_1 \, I_k \, x_3$

Unfortunately, the results displayed above lead us almost to impossibility results. We have considered different assumptions about individuals' rationality but, under every specification, the way of ensuring transitive collective relations approaches unanimous support. In other words, a high threshold is required in order to get transitive collective preference relations for every profile of individual reciprocal preference relations. In practice, an almost unanimous support is needed for avoiding intransitivities.

Indeed, transitivity may be considered as a really strong condition. We left the study of the conditions required to get acyclic collective preference relations as an alternative way to study the consistency of collective preference relations for further research. With regard to this, we want to remark the results in Kramer [32] and in Slutsky [45]. In the first case, acyclic collective preference relations are reached by using a specific qualified majority, the *minmax voting rule*, and a particular structure of dichotomous collective preferences, called *Type I utility functions*. In the second case, the collective preference relation is built on the basis of a majority which takes into account the logarithmic difference between the number of individuals who prefer an alternative, say x, to another, say y, and the number of individuals who prefer y to x. When this difference reaches a specific value, the collective preference relation is acyclic for any type of individual preferences (see also Craven [5] and Ferejohn and Grether [18]).

Moreover, we want to point out that there is a significant diparity between the possibility of having cycles and the empirical occurrence of them (see Gehrlein and Fishburn [21], Gehrlein [22] and Tangian [48], among others). We left the study of such occurrence in the case of $\widetilde{M}_k$ majorities for further research.

From a practical point of view, the class of majorities based on difference in support may be applied to several scenarios whenever one alternative has to be chosen with a desirable support with respect to other alternatives. Among international organizations, an interesting case is the European Commission, where voters are the different members states.

Acknowledgements

The authors are grateful to Jorge Alcalde-Unzu, Bernard De Baets and three anonymous referees for valuable suggestions and comments. This work is partially supported by the Junta de Castilla y León (Consejería de Educación y Cultura, Project GR99), the Spanish Ministry of Science and Innovation (Projects

ECO2009-07332 and ECO2009-12836), and ERDF.

References

[1] S. Alonso, F.J. Cabrerizo, F. Chiclana, F. Herrera, E. Herrera-Viedma, Group decision-making with incomplete fuzzy linguistic preference relations, International Journal of Intelligent Systems 24 (2009) 201–222.

[2] F. Chiclana, E. Herrera-Viedma, S. Alonso, F. Herrera, A note on the estimation of missing pairwise preference values: A uninorm consistency based method, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 16 (Suppl. 02) (2008) 19–32.

[3] F. Chiclana, E. Herrera-Viedma, S. Alonso, F. Herrera, Cardinal consistency of reciprocal preference relations: A characterization of multiplicative transitivity, IEEE Transactions on Fuzzy Systems 17 (2009) 14–23.

[4] M. de Condorcet, Essai sur l'Application de l'Analyse a la Probabilité des Decisions Rendues a la Probabilité des Voix, Imprimerie Royale, Paris, 1785.

[5] J. Craven, Majority voting and social choice, Review of Economic Studies 38 (1971) 265–267.

[6] M. Dasgupta, R. Deb, Transitivity and fuzzy preferences, Social Choice and Welfare 13 (1996) 305–318.

[7] P. Dasgupta, E. Maskin, Is majority rule the best voting method?, Mimeo (2003).

[8] B. De Baets, H. De Meyer, Transitivity frameworks for reciprocal relations: cycle-transitivity versus FG-transitivity, Fuzzy Sets and Systems 152 (2005) 249–270.

[9] B. De Baets, H. De Meyer, K. De Loof, On the cycle-transitivity of the mutual rank probability relation of a poset, Fuzzy Sets and Systems 161 (2010) 2695–2708.

[10] B. De Baets, H. De Meyer, B. De Schuymer, S. Jenei, Cyclic evaluation of transitivity of reciprocal relations, Social Choice and Welfare 26 (2006) 217–238.

[11] B. De Baets, B. Van de Walle, Weak and strong fuzzy interval orders, Fuzzy Sets and Systems 79 (1996) 213–225.

[12] B. De Baets, B. Van de Walle, E. Kerre, Fuzzy preference structures without incomparability, Fuzzy Sets and Systems 76 (1995) 333–348.

[13] S. Díaz, B. De Baets, S. Montes, Additive decomposition of fuzzy pre-orders, Fuzzy Sets and Systems 158 (2007) 830–842.

[14] S. Díaz, B. De Baets, S. Montes, On the compositional characterization of complete fuzzy pre-orders, Fuzzy Sets and Systems 159 (2008) 2221–2239.

[15] S. Díaz, S. Montes, B. De Baets, Transitive decomposition of fuzzy preference relations: the case of nilpotent minimum, Kybernetika 40 (2004) 71–88.

[16] S. Díaz, S. Montes, B. De Baets, Transitivity bounds in additive fuzzy preference structures, IEEE Transactions on Fuzzy Systems 15 (2007) 275–286.

[17] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.

[18] J.A. Ferejohn, D.M. Grether, On a class of rational social decisions procedures, Journal of Economic Theory 8 (1974) 471–482.

[19] P.C. Fishburn, The Theory of Social Choice, Princeton University Press, Princeton, 1973.

[20] P.C. Fishburn, Binary choice probabilities: on the varieties of stochastic transitivity, Journal of Mathematical Psychology 10 (1973) 327–352.

[21] W.V. Gehrlein, P.C. Fishburn, Condorcet's paradox and anonymous preference profiles, Public Choice 26 (1976) 1–18.

[22] W.V. Gehrlein, Probability calculations for transitivity of simple majority rule with anonymous voters, Public Choice 66 (1990) 253–259.

[23] J.L. García-Lapresta, B. Llamazares, Aggregation of fuzzy preferences: Some rules of the mean, Social Choice and Welfare 17 (2000) 673–690.

[24] J.L. García-Lapresta, B. Llamazares, Majority decisions based on difference of votes, Journal of Mathematical Economics 35 (2001) 463–481.

[25] J.L. García-Lapresta, B. Llamazares, Preference intensities and majority decisions based on difference of support between alternatives, Group Decision and Negotiation 19 (2010) 527–542.

[26] J.L. García-Lapresta, L.C. Meneses, Individual valued preferences and their aggregation: Analysis of consistency in a real case, Fuzzy Sets and Systems 151 (2005) 269–284.

[27] J.L. García-Lapresta, L.C. Meneses, Modelling rationality in a linguistic framework, Fuzzy Sets and Systems 160 (2009) 3211–3223.

[28] J.L. García-Lapresta, J. Montero, Consistency in preference modelling, in: B. Bouchon-Meunier, G. Coletti, R.R. Yager (Eds.), Modern Information Processing. From Theory to Applications, Elsevier, Amsterdam, 2006, pp. 87–97.

[29] E. Herrera-Viedma, F. Herrera, F. Chiclana, M. Luque, Some issues on consistency of fuzzy preference relations, European Journal of Operational Research 154 (2004) 98–109.

[30] N. Houy, Some further characterizations for the forgotten voting rules, Mathematical Social Sciences 53 (2007) 111–121.

[31] N. Jain, Transitivity of fuzzy relations and rational choice, Annals of Operations Research 23 (1990) 265–278.

[32] G.H. Kramer, A dynamical model of political equilibrium, Journal of Economic Theory 16 (1977) 310–334.

[33] B. Llamazares, Simple and absolute special majorities generated by OWA operators, European Journal of Operational Research 158 (2004) 707–720.

[34] B. Llamazares, The forgotten decision rules: Majority rules based on difference of votes, Mathematical Social Sciences 51 (2006) 311–326.

[35] B. Llamazares, Choosing OWA operator weights in the field of Social Choice, Information Sciences 177 (2007) 4745–4756.

[36] B. Llamazares, J.L. García-Lapresta, Voting systems generated by quasiarithmetic means and OWA operators, in: J. Fodor, B. De Baets (Eds.), Principles of Fuzzy Preference Modelling and Decision Making, Academia Press, Ghent, 2003, pp. 195–213.

[37] B. Llamazares, J.L. García-Lapresta, Extension of some voting systems to the field of gradual preferences, in: H. Bustince, F. Herrera, J. Montero (Eds.), Fuzzy Sets and Their Extensions: Representation, Aggregation and Models, Springer-Verlag, Berlin, 2008, pp. 292–310.

[38] K.O. May, A set of independent necessary and sufficient conditions for simple majority decisions, Econometrica 20 (1952) 680–684.

[39] I. McLean, A.B. Urken, Classics of Social Choice, University of Michigan Press, Ann Arbor, 1995.

[40] J.I. Morales, Memoria Matemática sobre el Cálculo de la Opinion en las Elecciones, Imprenta Real, Madrid, 1797.

[41] H. Nurmi, Approaches to collective decision making with fuzzy preference relations, Fuzzy Sets and Systems 6 (1981) 249–259.

[42] H. Nurmi, Fuzzy social choice: A selective retrospect, Soft Computing 12 (2008) 281–288.

[43] D.G. Saari, Consistency of decision processes, Annals of Operations Research 23 (1990) 103–137.

[44] A.K. Sen, Collective Choice and Social Welfare, Holden-Day, San Francisco, 1970.

[45] S. Slutsky, Equilibrium under α–majority voting, Econometrica 47 (1979) 1113–1125.

[46] Z. Switalski, Rationality of fuzzy reciprocal preference relations, Fuzzy Sets and Systems 107 (1999) 187–190.

[47] Z. Switalski, General transitivity conditions for fuzzy reciprocal preference matrices, Fuzzy Sets and Systems 137 (2003) 85–100.

[48] A.S. Tangian, Unlikelihood of Condorcet's paradox in a large society, Social Choice and Welfare 17 (2000) 337–365.

[49] T. Tanino, Fuzzy preference orderings in group decision making, Fuzzy Sets and Systems 12 (1984) 117–131.

[50] B. Van de Walle, B. De Baets, E. Kerre, Characterizable fuzzy preference structures, Annals of Operations Research 80 (1998) 105–136.

[51] L.A. Zadeh, Similarity relations and fuzzy orderings, Information Sciences 3 (1971) 177–200.