Research Article

A Study of SUOWA Operators in Two Dimensions

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SUOWA operators are a new class of aggregation functions that simultaneously generalize weighted means and OWA operators. They are Choquet integral-based operators with respect to normalized capacities; therefore, they possess some interesting properties such as continuity, monotonicity, idempotency, compensativeness and homogeneity of degree 1. In this paper we focus on two dimensions and show that any Choquet integral with respect to a normalized capacity can be expressed as a SUOWA operator.

1. Introduction

The study of aggregation operators has received special attention in the last years. This is due to the extensive applications of these functions for aggregating information in a wide variety of areas. Two of the best-known aggregation operators are the weighted means and the ordered weighted averaging (OWA) operators (Yager [1]). Both classes of functions are defined by means of weighting vectors, but their behavior is quite different: Weighted means allow to weight each information source in relation to their reliability while OWA operators allow to weight the values according to their ordering.

Although both families of operators allow to solve a wide range of problems, both weightings are necessary in some contexts. Some examples of these situations have been given by several authors (see, for instance, Torra [2, 3, 4], Torra and Godo [5, pp. 160–161], Torra and Narukawa [6, pp. 150–151], Roy [7], Yager and Alajlan [8], Llamazares [9] and the references therein) in fields as diverse as robotics, vision, fuzzy logic controllers, constraint satisfaction problems, scheduling, multicriteria aggregation problems and decision making.

A typical situation where both weighting are necessary is the following (Llamazares [9]): suppose we have several sensors to measure a physical property. On the one hand, sensors may be of different quality and precision, so a weighted mean type aggregation is necessary. On the other hand, to prevent a faulty sensor alters the measurement, we might consider an OWA type aggregation where the maximum and minimum values are not taking into account. A similar situation occurs when a committee of experts has to assess several candidates or proposals. On the one hand, a weighted mean type aggregation is suitable for reflecting the expertness or the confidence in the judgment of each expert. On the other hand, an OWA type aggregation allows us to deal with situations where an expert feels an excessive acceptance or rejection towards some of the candidates or proposals.

Different aggregation operators have appeared in the literature to deal with this kind of problems. A usual approach is to consider families of functions parametrized by two weighting vectors, one for the weighted mean and the other one for the OWA type aggregation, that generalize weighted means and OWA operators in the following sense: A weighted mean (or a OWA operator) is obtained when the other weighting vector has a "neutral" behavior; that is, it is $(1/n,\dots,1/n)$ (see Llamazares [10] for an analysis of some functions that generalize the weighted means and the OWA operators in this sense). Two of the solutions having better properties are the weighted OWA (WOWA) operator, proposed by Torra [3], and the semi-uninorm based ordered weighted averaging (SUOWA) operator, introduced by Llamazares [9].

The good properties of WOWA and SUOWA operators are due to they are Choquet integral-based operators with respect to normalized capacities. In the case of SUOWA operators, their capacities are the monotonic cover of certain games, which are defined by using the capacities associated with the weighted means and the OWA operators and "assembling" these values through semi-uninorms with neutral element $1/n$.

Because of their good properties, it seems interesting to analyze the behavior of SUOWA operators from different points of view. In this paper we consider the case of two dimensions that, although simple, it is attractive from a theoretical point of view, and we show that any Choquet integral with respect to a normalized capacity can be expressed as a SUOWA operator.

The remainder of the paper is organized as follows. In Section 2 we recall the concepts of semi-uninorm and uninorm and give some interesting examples of such functions. Section 3 is devoted to Choquet integral, including some of the most important particular cases: weighted means, OWA operators and SUOWA operators. In Section 4 we give the main results of the paper. Finally, some concluding remarks are provided in Section 5.

2. Semi-uninorms and uninorms

Throughout the paper we will use the following notation: $N = \{1,\dots,n\}$; given $A \subseteq N$, $|A|$ denotes the cardinality of A; vectors are denoted in bold and $\boldsymbol{\eta}$ denotes the tuple $(1/n,\dots,1/n) \in \mathbb{R}^n$. We write $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in N$. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, $[\cdot]$ and $(\cdot)$ denote permutations such that $x_{[1]} \geq \cdots \geq x_{[n]}$ and $x_{(1)} \leq \cdots \leq x_{(n)}$.

Semi-uninorms are a class of necessary functions in the definition of SUOWA operators. They are monotonic and have a neutral element in the interval $[0,1]$. These functions were introduced by Liu [11] as a generalization of uninorms, which, in turn, were proposed by Yager and Rybalov [12] as a generalization of t-norms and t-conorms.

Before introducing the concepts of semi-uninorm and uninorm, we recall some well-known properties of aggregation functions.

Definition 1. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function.

1. F is symmetric if $F(x_{\sigma(1)},\dots,x_{\sigma(n)}) = F(x_1,\dots,x_n)$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and for all permutation σ of N.
2. F is monotonic if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $F(\boldsymbol{x}) \geq F(\boldsymbol{y})$ for all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$.
3. F is idempotent if $F(x,\dots,x) = x$ for all $x \in \mathbb{R}$.
4. F is compensative (or internal) if $\min(\boldsymbol{x}) \leq F(\boldsymbol{x}) \leq \max(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
5. F is homogeneous of degree 1 (or ratio scale invariant) if $F(r\boldsymbol{x}) = rF(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and for all $r > 0$.

Definition 2. Let $U : [0,1]^2 \longrightarrow [0,1]$.

1. U is a semi-uninorm if it is monotonic and possesses a neutral element $e \in [0,1]$ ($U(e,x) = U(x,e) = x$ for all $x \in [0,1]$).
2. U is a uninorm if it is a symmetric and associative ($U(x,U(y,z)) = U(U(x,y),z)$ for all $x,y,z \in [0,1]$) semi-uninorm.

We denote by $\mathscr{U}^e$ (respectively, $\mathscr{U}_{\mathrm{i}}^e$) the set of semi-uninorms (respectively, idempotent semi-uninorms) with neutral element $e \in [0,1]$.

SUOWA operators are defined by using semi-uninorms with neutral element $1/n$. Moreover, they have to belong to the following subset (see Llamazares [9]):

$$\widetilde{\mathscr{U}}^{1/n} = \left\{ U \in \mathscr{U}^{1/n} \mid U(1/k,1/k) \leq 1/k \text{ for all } k \in N \right\}.$$

Obviously $\mathscr{U}_{\mathrm{i}}^{1/n} \subseteq \widetilde{\mathscr{U}}^{1/n}$. Notice that the smallest and the largest elements of $\widetilde{\mathscr{U}}^{1/n}$ are, respectively, the following semi-uninorms:

$$U_{\perp}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ 0 & \text{if } (x,y) \in [0,1/n)^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

$$U_{\top}(x,y) = \begin{cases} 1/k & \text{if } (x,y) \in I_k \setminus I_{k+1}, \text{where} \\ & I_k = \big(1/n, 1/k\big]^2 \ \big(k \in N\setminus\{n\}\big), \\ \min(x,y) & \text{if } (x,y) \in [0,1/n]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

In the case of idempotent semi-uninorms, the smallest and the largest elements of $\mathscr{U}_{\mathrm{i}}^{1/n}$ are, respectively, the following uninorms (which were given by Yager and Rybalov [12]):

$$U_{\min}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

$$U_{\max}(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,1/n]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

In addition to the previous ones, several procedures to construct semi-uninorms have been introduced by Llamazares [13]. One of them, which is based on ordinal sums of aggregation operators, allows us to get continuous semi-uninorms. Some of the most relevant continuous semi-uninorms obtained are the following:

$$U_{T_{\mathbf{L}}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ \max(x+y-1/n,0) & \text{otherwise.} \end{cases}$$

$$U_{\widetilde{P}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ nxy & \text{otherwise.} \end{cases}$$

$$U_{T_{\mathbf{M}}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ \min(x,y) & \text{if } (x,y) \in [0,1/n)^2, \\ x+y-1/n & \text{otherwise.} \end{cases}$$

$$U_{P}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ \min(x,y) & \text{if } (x,y) \in [0,1/n)^2, \\ nxy & \text{otherwise.} \end{cases}$$

Notice that the last two semi-uninorms are also idempotent. The plots of all these semi-uninorms are given, for the case $n = 4$, in Figures 1–8.

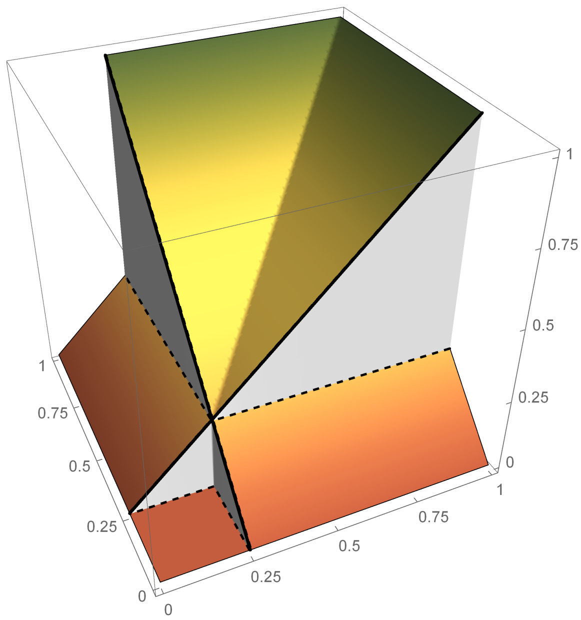

FIGURE 1: The semi-uninorm $U_{\perp}$ for $n = 4$.

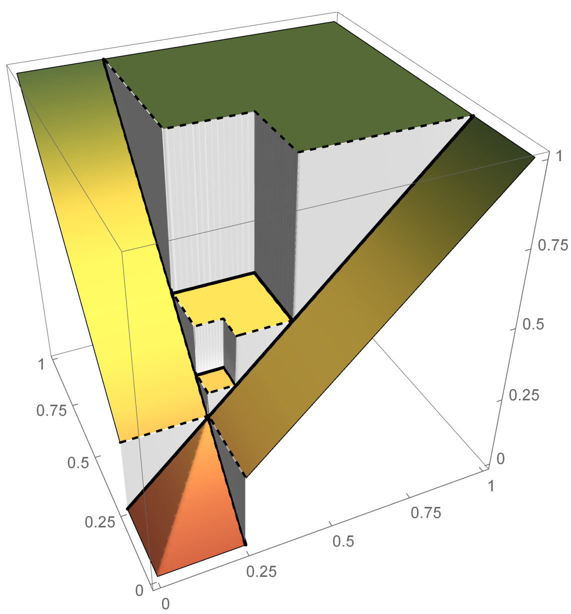

FIGURE 2: The semi-uninorm $U_{\top}$ for $n = 4$.

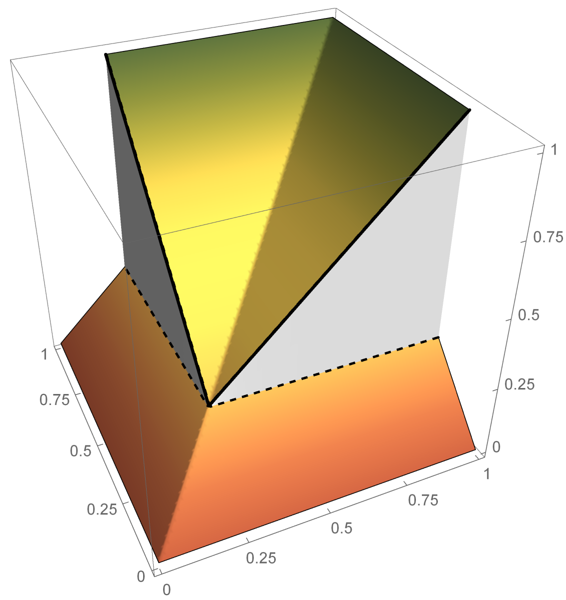

FIGURE 3: The uninorm $U_{\min}$ for $n = 4$.

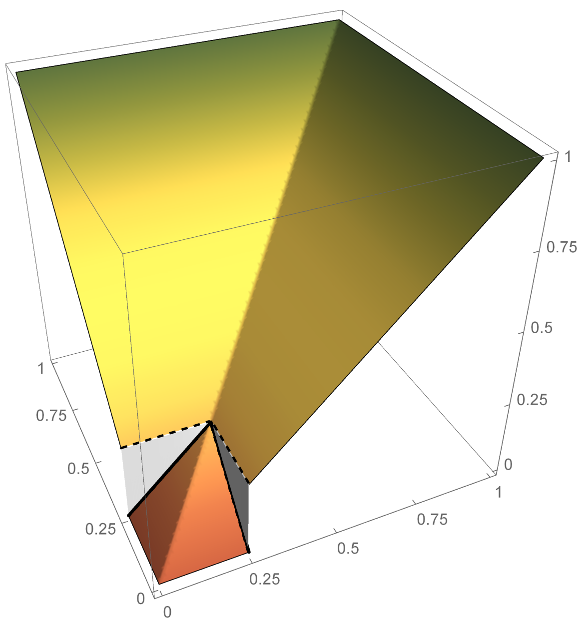

FIGURE 4: The uninorm $U_{\max}$ for $n = 4$.

3. Choquet integral

The notion of Choquet integral is based on that of capacity (see Choquet [14] and Murofushi and Sugeno [15]). The concept of capacity resembles that of probability measure but in the definition of the former additivity is replaced by monotonicity (see also fuzzy measures in Sugeno [16]). A game is then a generalization of a capacity where the monotonicity is no longer required.

Definition 3.

1. A game υ on N is a set function, $\upsilon : 2^N \longrightarrow \mathbb{R}$ satisfying $\upsilon(\varnothing) = 0$.
2. A capacity (or fuzzy measure) μ on N is a game on N satisfying $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$. In particular, it follows that $\mu : 2^N \longrightarrow [0, \infty)$. A capacity μ is said to be normalized if $\mu(N) = 1$.

A straightforward way to get a capacity from a game is to consider the monotonic cover of the game (see Maschler and Peleg [17] and Maschler *et al.* [18]).

Definition 4. Let υ be a game on N. The monotonic cover

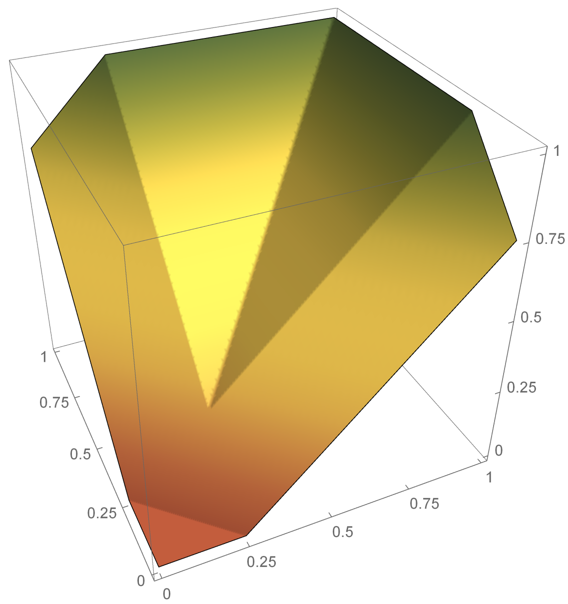

FIGURE 5: The uninorm $U_{T_{\mathrm{L}}}$ for $n = 4$.

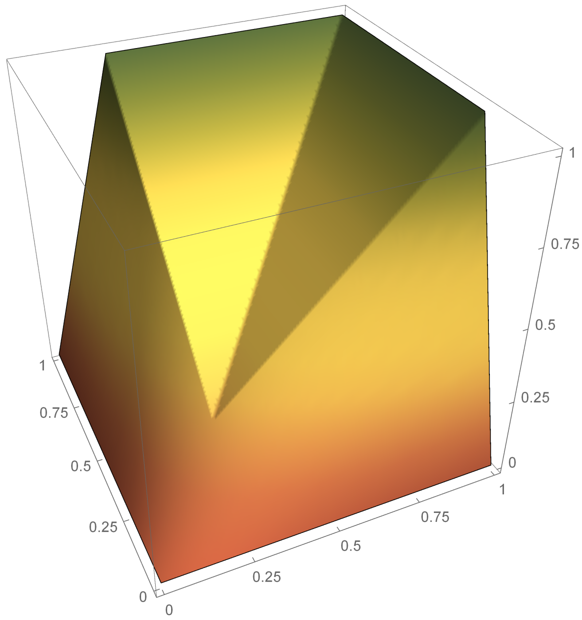

FIGURE 6: The uninorm $U_{\tilde{P}}$ for $n = 4$.

of υ is the set function $\hat{\upsilon}$ given by

$$\hat{\upsilon}(A) = \max_{B \subseteq A} \upsilon(B).$$

Some basic properties of $\hat{\upsilon}$ are given in the sequel.

Remark 1. Let υ be a game on N. Then:

1. $\hat{\upsilon}$ is a capacity.
2. If υ is a capacity, then $\hat{\upsilon} = \upsilon$.
3. If $\upsilon(A) \leq 1$ for all $A \subseteq N$ and $\upsilon(N) = 1$, then $\hat{\upsilon}$ is a normalized capacity.

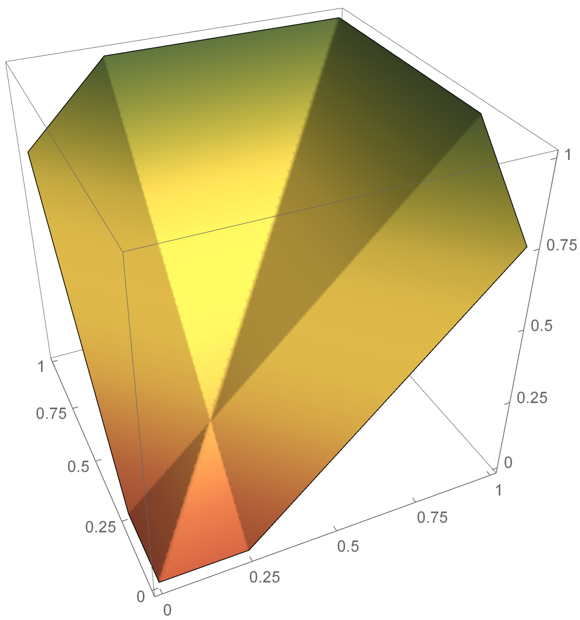

FIGURE 7: The uninorm $U_{T_{\mathrm{M}}}$ for $n = 4$.

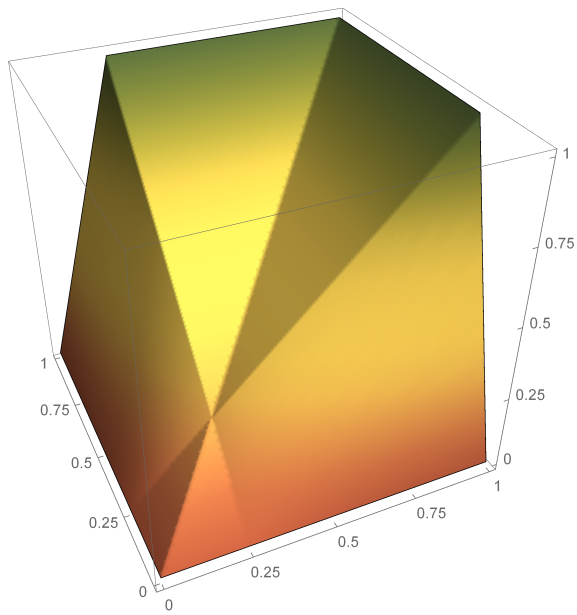

FIGURE 8: The uninorm U_P for $n = 4$.

Although the Choquet integral is usually defined as a functional (see, for instance, Choquet [14], Murofushi and Sugeno [15] and Denneberg [19]), in this paper we consider the Choquet integral as an aggregation function over $\mathbb{R}^n$ (see, for instance, Grabisch *et al.* [20, p. 181]). Moreover, we define the Choquet integral for all vectors of $\mathbb{R}^n$ instead of nonnegative vectors given that we are actually considering the asymmetric Choquet integral with respect to μ (on this, see again Grabisch *et al.* [20, p. 182]).

Definition 5. Let μ be a capacity on N. The Choquet integral with respect to μ is the function $\mathscr{C}_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\mathscr{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu(B_{(i)})\big(x_{(i)} - x_{(i-1)}\big),$$

where $B_{(i)} = \{(i), \ldots, (n)\}$, and we use the convention $x_{(0)} = 0$.

It is worth noting that the Choquet integral has several properties which are useful in certain information aggregation contexts (see, for instance, Grabisch *et al.* [20, pp. 192–193 and p. 196]).

Remark 2. Let μ be a capacity on N. Then $\mathscr{C}_\mu$ is continuous, monotonic and homogeneous of degree 1. Moreover, it is idempotent and compensative when μ is a normalized capacity.

Notice that the Choquet integral can also be represented by using a decreasing sequences of values (see, for instance, Torra [21] and Llamazares [9]):

$$\mathscr{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu(A_{[i]})\big(x_{[i]} - x_{[i+1]}\big) \tag{1}$$

where $A_{[i]} = \{[1], \ldots, [i]\}$, and we use the convention $x_{[n+1]} = 0$.

From the previous expression, it is straightforward to show explicitly the weights of the values $x_{[i]}$ by representing the Choquet integral as follows:

$$\mathscr{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \big(\mu(A_{[i]}) - \mu(A_{[i-1]})\big)x_{[i]},$$

where we use the convention $A_{[0]} = \varnothing$.

3.1 Weighted means and OWA operators

Weighted means and OWA operators (Yager [1]) are well-known functions in the field of aggregation operators. Both families of functions are defined in terms of weight distributions that add up to 1.

Definition 6. A vector $\boldsymbol{q} \in \mathbb{R}^n$ is a weighting vector if $\boldsymbol{q} \in [0,1]^n$ and $\sum_{i=1}^{n} q_i = 1$.

The set of all weighting vectors of $\mathbb{R}^n$ will be denoted by $\mathscr{W}_n$.

Definition 7. Let $\boldsymbol{p}$ be a weighting vector. The weighted mean associated with $\boldsymbol{p}$ is the function $M_{\boldsymbol{p}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$M_{\boldsymbol{p}}(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i.$$

Definition 8. Let $\boldsymbol{w}$ be a weighting vector. The OWA operator associated with $\boldsymbol{w}$ is the function $O_{\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$O_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{[i]}.$$

It is well known that weighted means and OWA operators are a special type of Choquet integral (see, for instance, Fodor *et al.* [22], Grabisch [23, 24] or Llamazares [9]).

Remark 3.

1. If $\boldsymbol{p}$ is a weighting vector, then the weighted mean $M_{\boldsymbol{p}}$ is the Choquet integral with respect to the normalized capacity $\mu_{\boldsymbol{p}}(A) = \sum_{i \in A} p_i$.
2. If $\boldsymbol{w}$ is a weighting vector, then the OWA operator $O_{\boldsymbol{w}}$ is the Choquet integral with respect to the normalized capacity $\mu_{|\boldsymbol{w}|}(A) = \sum_{i=1}^{|A|} w_i$.

So, according to Remark 2, weighted means and OWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1. Moreover, in the case of OWA operators, given that the values of the variables are previously ordered in a decreasing way, they are also symmetric.

3.2 SUOWA operators

SUOWA operators were introduced by Llamazares [9] in order to consider situations where both the importance of information sources and the importance of values had to be taken into account. These functions are Choquet integral-based operators where their capacities are the monotonic cover of certain games. These games are defined by using semi-uninorms with neutral element $1/n$ and the values of the capacities associated with the weighted means and the OWA operators. To be specific, the games from which SUOWA operators are built are defined as follows.

Definition 9. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let $U \in \widetilde{\mathscr{U}}^{1/n}$.

1. The game associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U is the set function $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}} : 2^N \longrightarrow \mathbb{R}$ defined by
$$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) = |A|\, U\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, \frac{\mu_{|\boldsymbol{w}|}(A)}{|A|}\right) = |A|\, U\left(\frac{\sum_{i\in A} p_i}{|A|}, \frac{\sum_{i=1}^{|A|} w_i}{|A|}\right)$$
if $A \neq \varnothing$, and $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(\varnothing) = 0$.
2. $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$, the monotonic cover of the game $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}$, will be called the capacity associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U.

Notice that $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(N) = 1$. Moreover, since $U \in \widetilde{\mathscr{U}}^{1/n}$ we have $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) \leq 1$ for all $A \subseteq N$ (see Llamazares [9]). Therefore, according to the third item of Remark 1, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is always a normalized capacity.

Definition 10. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let $U \in \widetilde{\mathscr{U}}^{1/n}$. The SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U is the function $S^U_{\boldsymbol{p},\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} s_i x_{[i]},$$

where $s_i = \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]}) - \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i-1]})$ for all $i \in N$, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is the capacity associated with $\boldsymbol{p},\boldsymbol{w}$ and U, and $A_{[i]} = \{[1],\dots,[i]\}$ (with the convention that $A_{[0]} = \varnothing$).

According to expression (1), the SUOWA operator associated with $\boldsymbol{p},\boldsymbol{w}$ and U can also be written as

$$S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]})\big(x_{[i]} - x_{[i+1]}\big).$$

By the choice of $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ we have $S^U_{\boldsymbol{p},\boldsymbol{\eta}} = M_{\boldsymbol{p}}$ and $S^U_{\boldsymbol{\eta},\boldsymbol{w}} = O_{\boldsymbol{w}}$ for any $U \in \widetilde{\mathscr{U}}^{1/n}$. Moreover, by Remark 2 and given that $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity, SUOWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1.

4. The results

The use of Choquet integral has become more and more extensive in the last years (see, for instance, Grabisch *et al.* [25] and Grabisch and Labreuche [26]). Although simple, the case $n = 2$ is interesting from a theoretical point of view. Thus, for instance, Grabisch *et al.* [20, p. 204] show that, in this case, any Choquet integral with respect to a normalized capacity can be written as a convex combination of a minimum, a maximum and two projections; that is, given a normalized capacity μ, there exists a weighting vector $\boldsymbol{\lambda}$ belonging to $\mathscr{W}_4$ such that

$$\mathscr{C}_\mu(x_1,x_2) = \lambda_1 \min(x_1,x_2) + \lambda_2 \max(x_1,x_2) + \lambda_3 x_1 + \lambda_4 x_2.$$

In our case we are going to show that any Choquet integral with respect to a normalized capacity can be written as a SUOWA operator. Notice that, when $n = 2$, $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}$ is always a normalized capacity for any weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ and for any semi-uninorm U. Therefore, given a normalized capacity μ, we need to prove that there exist weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ and a semi-uninorm U such that

$$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(\{1\}) = U(p_1,w_1) = \mu_1,$$
$$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(\{2\}) = U(p_2,w_1) = \mu_2,$$

where we use the notation μ_1 and μ_2 to denote the values $\mu(\{1\})$ and $\mu(\{2\})$, respectively.

Firstly we are going to show that in the case of the semi-uninorms $U_\perp$, $U_\top$, $U_{\min}$ and $U_{\max}$, there exist normalized capacities which cannot be expressed as SUOWA operators. For this, we will use the following lemma.

Lemma 1. *If $U \in \{U_\perp, U_\top, U_{\min}, U_{\max}\}$, then $U(x,y) = 0.5$ if and only if $x = y = 0.5$.*

Proof. Let $U \in \{U_\perp, U_\top, U_{\min}, U_{\max}\}$. Since 0.5 is the neutral element of U, we have $U(0.5,0.5) = 0.5$.

Conversely, suppose $U(x,y) = 0.5$. In Table 1, where 0.5^- stands for a value that belongs to $[0,0.5)$ and 0.5^+ stands for a value that belongs to $(0.5,1]$, we show the values taken by the semi-uninorms $U_\perp$, $U_{\min}$, $U_{\max}$, and $U_\top$ when $(x,y) \in [0,1]^2$. Therefore, if $U(x,y) = 0.5$, then necessarily $x = y = 0.5$. □

TABLE 1: Values taken by $U_\perp$, $U_{\min}$, $U_{\max}$, and $U_\top$.

x	y	$U_\perp(x,y)$	$U_{\min}(x,y)$	$U_{\max}(x,y)$	$U_\top(x,y)$
0.5^-	0.5^-	0	$\min(x,y)$	$\min(x,y)$	$\min(x,y)$
0.5^-	0.5	x	x	x	x
0.5^-	0.5^+	x	x	y	y
0.5	0.5^-	y	y	y	y
0.5	0.5	0.5	0.5	0.5	0.5
0.5	0.5^+	y	y	y	y
0.5^+	0.5^-	y	y	x	x
0.5^+	0.5	x	x	x	x
0.5^+	0.5^+	$\max(x,y)$	$\max(x,y)$	$\max(x,y)$	1

Theorem 1. *Let μ be the normalized capacity on $N = \{1,2\}$ such that $\mu_1 = 0$ and $\mu_2 = 0.5$. If $U \in \{U_\perp, U_\top, U_{\min}, U_{\max}\}$, then there do not exist weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ such that $\mu = \upsilon^U_{\boldsymbol{p},\boldsymbol{w}}$.*

Proof. Given $U \in \{U_\perp, U_\top, U_{\min}, U_{\max}\}$, consider two weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ such that $U(p_2,w_1) = 0.5$. By Lemma 1 we have $p_2 = w_1 = 0.5$. Therefore, $U(p_1,w_1) = U(0.5,0.5) = 0.5$ and, consequently, $U(p_1,w_1) = 0$ is not possible. □

In each of the following theorems we consider the semi-uninorms $U_{T_{\mathbf{L}}}$, $U_{T_{\mathbf{M}}}$, $U_{\widetilde{P}}$ and U_P, respectively, and we show that any normalized capacity can be written as a SUOWA operator associated with appropriate weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$, which are given explicitly.

Theorem 2. *Let μ be a normalized capacity on $N = \{1,2\}$ and let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors defined as follows:*

1. *If $\mu_1 + \mu_2 < 1$, then*

$$\boldsymbol{p} = \left(0.5 + \frac{\mu_1 - \mu_2}{2}, 0.5 + \frac{\mu_2 - \mu_1}{2}\right),$$
$$\boldsymbol{w} = \left(\frac{\mu_1 + \mu_2}{2}, 1 - \frac{\mu_1 + \mu_2}{2}\right).$$

2. *If $\mu_1 + \mu_2 \geq 1$ and $\min(\mu_1,\mu_2) < 0.5$, then*

$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 > 0.5, \\ (1-\mu_2, \mu_2) & \text{if } \mu_2 > 0.5, \end{cases}$$
$$\boldsymbol{w} = \big(\mu_1 + \mu_2 - 0.5, 1.5 - \mu_1 - \mu_2\big).$$

3. *If $\min(\mu_1,\mu_2) \geq 0.5$, then*

$$\boldsymbol{p} = \big(0.5 + \mu_1 - \mu_2, 0.5 + \mu_2 - \mu_1\big),$$
$$\boldsymbol{w} = \big(\max(\mu_1,\mu_2), 1 - \max(\mu_1,\mu_2)\big).$$

Then $\mu = \upsilon^{U_{T_{\mathbf{L}}}}_{\boldsymbol{p},\boldsymbol{w}}$; that is, $\mathscr{C}_\mu = S^{U_{T_{\mathbf{L}}}}_{\boldsymbol{p},\boldsymbol{w}}$.

Proof. Let μ be a normalized capacity on $N = \{1,2\}$ and recall that, when $n = 2$, the semi-uninorm $U_{T_{\mathbf{L}}}$ is defined by

$$U_{T_{\mathbf{L}}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [0.5,1]^2, \\ \max(x+y-0.5,0) & \text{otherwise.} \end{cases}$$

We distinguish the following cases:

1. If $\mu_1 + \mu_2 < 1$, consider

$$\boldsymbol{p} = \left(0.5 + \frac{\mu_1 - \mu_2}{2}, 0.5 + \frac{\mu_2 - \mu_1}{2}\right),$$
$$\boldsymbol{w} = \left(\frac{\mu_1 + \mu_2}{2}, 1 - \frac{\mu_1 + \mu_2}{2}\right).$$

Then,

$$U_{T_{\mathbf{L}}}(p_1,w_1) = 0.5 + \frac{\mu_1 - \mu_2}{2} + \frac{\mu_1 + \mu_2}{2} - 0.5 = \mu_1,$$
$$U_{T_{\mathbf{L}}}(p_2,w_1) = 0.5 + \frac{\mu_2 - \mu_1}{2} + \frac{\mu_1 + \mu_2}{2} - 0.5 = \mu_2.$$

2. If $\mu_1 + \mu_2 \geq 1$ and $\min(\mu_1,\mu_2) < 0.5$, consider

$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 > 0.5, \\ (1-\mu_2, \mu_2) & \text{if } \mu_2 > 0.5, \end{cases}$$
$$\boldsymbol{w} = \big(\mu_1 + \mu_2 - 0.5, 1.5 - \mu_1 - \mu_2\big).$$

We distinguish two cases:

(a) If $\mu_1 > 0.5$, then

$$U_{T_{\mathbf{L}}}(p_1,w_1) = \max\big(\mu_1, \mu_1 + \mu_2 - 0.5\big) = \mu_1,$$
$$U_{T_{\mathbf{L}}}(p_2,w_1) = 1 - \mu_1 + \mu_1 + \mu_2 - 0.5 - 0.5 = \mu_2.$$

(b) If $\mu_2 > 0.5$, then

$$U_{T_{\mathbf{L}}}(p_1,w_1) = 1 - \mu_2 + \mu_1 + \mu_2 - 0.5 - 0.5 = \mu_1,$$
$$U_{T_{\mathbf{L}}}(p_2,w_1) = \max\big(\mu_2, \mu_1 + \mu_2 - 0.5\big) = \mu_2.$$

3. If $\min(\mu_1,\mu_2) \geq 0.5$, consider

$$\boldsymbol{p} = \big(0.5 + \mu_1 - \mu_2, 0.5 + \mu_2 - \mu_1\big),$$
$$\boldsymbol{w} = \big(\max(\mu_1,\mu_2), 1 - \max(\mu_1,\mu_2)\big).$$

We distinguish three cases:

(a) If $\mu_1 > \mu_2$, then

$$U_{T_{\mathbf{L}}}(p_1,w_1) = \max\big(0.5 + \mu_1 - \mu_2, \mu_1\big) = \mu_1,$$
$$U_{T_{\mathbf{L}}}(p_2,w_1) = 0.5 + \mu_2 - \mu_1 + \mu_1 - 0.5 = \mu_2.$$

(b) If $\mu_1 = \mu_2$, then

$$U_{T_{\mathbf{L}}}(p_1,w_1) = \max\big(0.5, \mu_1\big) = \mu_1,$$
$$U_{T_{\mathbf{L}}}(p_2,w_1) = \max\big(0.5, \mu_2\big) = \mu_2.$$

(c) If $\mu_1 < \mu_2$, then

$$U_{T_{\mathbf{L}}}(p_1,w_1) = 0.5 + \mu_1 - \mu_2 + \mu_2 - 0.5 = \mu_1,$$
$$U_{T_{\mathbf{L}}}(p_2,w_1) = \max\big(0.5 + \mu_2 - \mu_1, \mu_2\big) = \mu_2. \quad \square$$

Theorem 3. *Let μ be a normalized capacity on $N = \{1,2\}$ and let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors defined as follows:*

1. *If $\max(\mu_1,\mu_2) < 0.5$, then*

$$\boldsymbol{p} = \big(0.5 + \mu_1 - \mu_2, 0.5 + \mu_2 - \mu_1\big),$$
$$\boldsymbol{w} = \big(\min(\mu_1,\mu_2), 1 - \min(\mu_1,\mu_2)\big).$$

2. *If $\mu_1 + \mu_2 < 1$ and $\max(\mu_1,\mu_2) \geq 0.5$, then*

$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 < 0.5, \\ (1-\mu_2, \mu_2) & \text{if } \mu_2 < 0.5, \end{cases}$$
$$\boldsymbol{w} = \big(\mu_1 + \mu_2 - 0.5, 1.5 - \mu_1 - \mu_2\big).$$

3. *If $\mu_1 + \mu_2 \geq 1$ and $\min(\mu_1,\mu_2) < 0.5$, then*

$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 > 0.5, \\ (1-\mu_2, \mu_2) & \text{if } \mu_2 > 0.5, \end{cases}$$
$$\boldsymbol{w} = \big(\mu_1 + \mu_2 - 0.5, 1.5 - \mu_1 - \mu_2\big).$$

4. *If $\min(\mu_1,\mu_2) \geq 0.5$, then*

$$\boldsymbol{p} = \big(0.5 + \mu_1 - \mu_2, 0.5 + \mu_2 - \mu_1\big),$$
$$\boldsymbol{w} = \big(\max(\mu_1,\mu_2), 1 - \max(\mu_1,\mu_2)\big).$$

Then $\mu = \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{M}}}}$; that is, $\mathscr{C}_\mu = S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{M}}}}$.

Proof. Let μ be a normalized capacity on $N = \{1,2\}$ and recall that, when $n = 2$, the semi-uninorm $U_{T_{\mathbf{M}}}$ is defined by

$$U_{T_{\mathbf{M}}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [0.5,1]^2, \\ \min(x,y) & \text{if } (x,y) \in [0,0.5)^2, \\ x+y-0.5 & \text{otherwise.} \end{cases}$$

We distinguish the following cases:

1. If $\max(\mu_1,\mu_2) < 0.5$, consider

$$\boldsymbol{p} = \big(0.5 + \mu_1 - \mu_2, 0.5 + \mu_2 - \mu_1\big),$$
$$\boldsymbol{w} = \big(\min(\mu_1,\mu_2), 1 - \min(\mu_1,\mu_2)\big).$$

We distinguish three cases:

(a) If $\mu_1 < \mu_2$, then

$$U_{T_{\mathbf{M}}}(p_1,w_1) = \min\big(0.5 + \mu_1 - \mu_2, \mu_1\big) = \mu_1,$$
$$U_{T_{\mathbf{M}}}(p_2,w_1) = 0.5 + \mu_2 - \mu_1 + \mu_1 - 0.5 = \mu_2.$$

(b) If $\mu_1 = \mu_2$, then

$$U_{T_{\mathbf{M}}}(p_1,w_1) = \min\big(0.5, \mu_1\big) = \mu_1,$$
$$U_{T_{\mathbf{M}}}(p_2,w_1) = \min\big(0.5, \mu_2\big) = \mu_2.$$

(c) If $\mu_1 > \mu_2$, then

$$U_{T_\mathbf{M}}(p_1,w_1) = 0.5+\mu_1-\mu_2+\mu_2-0.5 = \mu_1,$$
$$U_{T_\mathbf{M}}(p_2,w_1) = \min\big(0.5+\mu_2-\mu_1,\mu_2\big) = \mu_2.$$

2. If $\mu_1+\mu_2<1$ and $\max(\mu_1,\mu_2)\geq 0.5$, consider

$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 < 0.5, \\ (1-\mu_2,\mu_2) & \text{if } \mu_2 < 0.5, \end{cases}$$
$$\boldsymbol{w} = \big(\mu_1+\mu_2-0.5, 1.5-\mu_1-\mu_2\big).$$

We distinguish two cases:

(a) If $\mu_1 < 0.5$, then

$$U_{T_\mathbf{M}}(p_1,w_1) = \min\big(\mu_1,\mu_1+\mu_2-0.5\big) = \mu_1,$$
$$U_{T_\mathbf{M}}(p_2,w_1) = 1-\mu_1+\mu_1+\mu_2-0.5-0.5 = \mu_2.$$

(b) If $\mu_2 < 0.5$, then

$$U_{T_\mathbf{M}}(p_1,w_1) = 1-\mu_2+\mu_1+\mu_2-0.5-0.5 = \mu_1,$$
$$U_{T_\mathbf{M}}(p_2,w_1) = \min\big(\mu_2,\mu_1+\mu_2-0.5\big) = \mu_2.$$

3. If $\mu_1+\mu_2\geq 1$ and $\min(\mu_1,\mu_2)<0.5$, then the proof of this case is similar to that of the second item in Theorem 2.

4. If $\min(\mu_1,\mu_2)\geq 0.5$, then the proof of this case is similar to that of the third item in Theorem 2. □

Theorem 4. *Let μ be a normalized capacity on $N=\{1,2\}$ and let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors defined as follows:*

1. If $\mu_1+\mu_2<1$, then

$$\boldsymbol{p} = \begin{cases} (p_1,p_2)\in\mathscr{W}_2 & \text{if } \mu_1=\mu_2=0, \\ \left(\dfrac{\mu_1}{\mu_1+\mu_2},\dfrac{\mu_2}{\mu_1+\mu_2}\right) & \text{otherwise,} \end{cases}$$
$$\boldsymbol{w} = \left(\frac{\mu_1+\mu_2}{2}, 1-\frac{\mu_1+\mu_2}{2}\right).$$

2. If $\mu_1+\mu_2\geq 1$ and $\min(\mu_1,\mu_2)<2\max(\mu_1,\mu_2)\big(1-\max(\mu_1,\mu_2)\big)$, then

$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 \geq \mu_2, \\ (1-\mu_2,\mu_2) & \text{if } \mu_1 < \mu_2, \end{cases}$$
$$\boldsymbol{w} = \left(\frac{\min(\mu_1,\mu_2)}{2\big(1-\max(\mu_1,\mu_2)\big)}, 1-\frac{\min(\mu_1,\mu_2)}{2\big(1-\max(\mu_1,\mu_2)\big)}\right).$$

3. If $\mu_1+\mu_2\geq 1$ and $\min(\mu_1,\mu_2)\geq 2\max(\mu_1,\mu_2)\big(1-\max(\mu_1,\mu_2)\big)$, then

$$\boldsymbol{p} = \begin{cases} \left(1-\dfrac{\mu_2}{2\mu_1},\dfrac{\mu_2}{2\mu_1}\right) & \text{if } \mu_1 \geq \mu_2, \\ \left(\dfrac{\mu_1}{2\mu_2},1-\dfrac{\mu_1}{2\mu_2}\right) & \text{if } \mu_1 < \mu_2, \end{cases}$$
$$\boldsymbol{w} = \big(\max(\mu_1,\mu_2), 1-\max(\mu_1,\mu_2)\big).$$

Then $\mu = \upsilon^{U_{\tilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}$; that is, $\mathscr{C}_\mu = S^{U_{\tilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}$.

Proof. Let μ be a normalized capacity on $N=\{1,2\}$ and recall that, when $n=2$, the semi-uninorm $U_{\tilde{P}}$ is defined by

$$U_{\tilde{P}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y)\in[0.5,1]^2, \\ 2xy & \text{otherwise.} \end{cases}$$

We distinguish the following cases:

1. If $\mu_1+\mu_2<1$, consider

$$\boldsymbol{p} = \begin{cases} (p_1,p_2)\in\mathscr{W}_2 & \text{if } \mu_1=\mu_2=0, \\ \left(\dfrac{\mu_1}{\mu_1+\mu_2},\dfrac{\mu_2}{\mu_1+\mu_2}\right) & \text{otherwise,} \end{cases}$$
$$\boldsymbol{w} = \left(\frac{\mu_1+\mu_2}{2}, 1-\frac{\mu_1+\mu_2}{2}\right).$$

We distinguish two cases:

(a) If $\mu_1=\mu_2=0$, then

$$U_{\tilde{P}}(p_1,w_1) = 2\cdot p_1\cdot 0 = 0 = \mu_1,$$
$$U_{\tilde{P}}(p_2,w_1) = 2\cdot p_2\cdot 0 = 0 = \mu_2.$$

(b) If $(\mu_1,\mu_2)\neq(0,0)$, then

$$U_{\tilde{P}}(p_1,w_1) = 2\frac{\mu_1}{\mu_1+\mu_2}\frac{\mu_1+\mu_2}{2} = \mu_1,$$
$$U_{\tilde{P}}(p_2,w_1) = 2\frac{\mu_2}{\mu_1+\mu_2}\frac{\mu_1+\mu_2}{2} = \mu_2.$$

2. If $\mu_1+\mu_2\geq 1$ and $\min(\mu_1,\mu_2)<2\max(\mu_1,\mu_2)\big(1-\max(\mu_1,\mu_2)\big)$, then notice that the case $\max(\mu_1,\mu_2)=1$ is not possible. Moreover, we have

$$\frac{\min(\mu_1,\mu_2)}{2\big(1-\max(\mu_1,\mu_2)\big)} < \max(\mu_1,\mu_2).$$

On the other hand, given that $\min(\mu_1,\mu_2)\geq 1-\max(\mu_1,\mu_2)$, we get

$$\frac{\min(\mu_1,\mu_2)}{2\big(1-\max(\mu_1,\mu_2)\big)} \geq 0.5,$$

and, consequently, $\max(\mu_1,\mu_2)>0.5$. Now consider the following weighting vectors:

$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 \geq \mu_2, \\ (1-\mu_2,\mu_2) & \text{if } \mu_1 < \mu_2, \end{cases}$$
$$\boldsymbol{w} = \left(\frac{\min(\mu_1,\mu_2)}{2\big(1-\max(\mu_1,\mu_2)\big)}, 1-\frac{\min(\mu_1,\mu_2)}{2\big(1-\max(\mu_1,\mu_2)\big)}\right).$$

We distinguish two cases:

(a) If $\mu_1 \geq \mu_2$, then

$$U_{\widetilde{P}}(p_1, w_1) = \max\left(\mu_1, \frac{\mu_2}{2(1-\mu_1)}\right) = \mu_1,$$
$$U_{\widetilde{P}}(p_2, w_1) = 2(1-\mu_1)\frac{\mu_2}{2(1-\mu_1)} = \mu_2.$$

(b) If $\mu_1 < \mu_2$, then

$$U_{\widetilde{P}}(p_1, w_1) = 2(1-\mu_2)\frac{\mu_1}{2(1-\mu_2)} = \mu_1,$$
$$U_{\widetilde{P}}(p_2, w_1) = \max\left(\mu_2, \frac{\mu_1}{2(1-\mu_2)}\right) = \mu_2.$$

3. If $\mu_1 + \mu_2 \geq 1$ and $\min(\mu_1,\mu_2) \geq 2\max(\mu_1,\mu_2)\big(1 - \max(\mu_1,\mu_2)\big)$, then $\max(\mu_1,\mu_2) \geq 0.5$, and we also have
$$\frac{\min(\mu_1,\mu_2)}{2\max(\mu_1,\mu_2)} \geq 1 - \max(\mu_1,\mu_2),$$
or, equivalently,
$$1 - \frac{\min(\mu_1,\mu_2)}{2\max(\mu_1,\mu_2)} \leq \max(\mu_1,\mu_2).$$
On the other hand, since $\min(\mu_1,\mu_2) \leq \max(\mu_1,\mu_2)$, we get
$$\frac{\min(\mu_1,\mu_2)}{2\max(\mu_1,\mu_2)} \leq 0.5,$$
and, consequently,
$$1 - \frac{\min(\mu_1,\mu_2)}{2\max(\mu_1,\mu_2)} \geq 0.5.$$

Consider now the following weighting vectors:
$$\boldsymbol{p} = \begin{cases} \left(1 - \frac{\mu_2}{2\mu_1}, \frac{\mu_2}{2\mu_1}\right) & \text{if } \mu_1 \geq \mu_2, \\ \left(\frac{\mu_1}{2\mu_2}, 1 - \frac{\mu_1}{2\mu_2}\right) & \text{if } \mu_1 < \mu_2, \end{cases}$$
$$\boldsymbol{w} = \big(\max(\mu_1,\mu_2), 1 - \max(\mu_1,\mu_2)\big).$$

We distinguish three cases:

(a) If $\mu_1 > \mu_2$, then

$$U_{\widetilde{P}}(p_1, w_1) = \max\left(1 - \frac{\mu_2}{2\mu_1}, \mu_1\right) = \mu_1,$$
$$U_{\widetilde{P}}(p_2, w_1) = 2\frac{\mu_2}{2\mu_1}\mu_1 = \mu_2.$$

(b) If $\mu_1 = \mu_2$, then

$$U_{\widetilde{P}}(p_1, w_1) = \max(0.5, \mu_1) = \mu_1,$$
$$U_{\widetilde{P}}(p_2, w_1) = \max(0.5, \mu_2) = \mu_2.$$

(c) If $\mu_1 < \mu_2$, then

$$U_{\widetilde{P}}(p_1, w_1) = 2\frac{\mu_1}{2\mu_2}\mu_2 = \mu_1,$$
$$U_{\widetilde{P}}(p_2, w_1) = \max\left(1 - \frac{\mu_1}{2\mu_2}, \mu_2\right) = \mu_2. \quad \square$$

Theorem 5. *Let μ be a normalized capacity on $N = \{1,2\}$ and let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors defined as follows:*

1. *If $\mu_1 + \mu_2 < 1$ and $\max(\mu_1,\mu_2) < 2\min(\mu_1,\mu_2)\big(1 - \min(\mu_1,\mu_2)\big)$, then*
$$\boldsymbol{p} = \begin{cases} \left(1 - \frac{\mu_2}{2\mu_1}, \frac{\mu_2}{2\mu_1}\right) & \text{if } \mu_1 \leq \mu_2, \\ \left(\frac{\mu_1}{2\mu_2}, 1 - \frac{\mu_1}{2\mu_2}\right) & \text{if } \mu_1 > \mu_2, \end{cases}$$
$$\boldsymbol{w} = \big(\min(\mu_1,\mu_2), 1 - \min(\mu_1,\mu_2)\big).$$

2. *If $\mu_1 + \mu_2 < 1$ and $\max(\mu_1,\mu_2) \geq 2\min(\mu_1,\mu_2)\big(1 - \min(\mu_1,\mu_2)\big)$, then*
$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 \leq \mu_2, \\ (1-\mu_2, \mu_2) & \text{if } \mu_1 > \mu_2, \end{cases}$$
$$\boldsymbol{w} = \left(\frac{\max(\mu_1,\mu_2)}{2\big(1-\min(\mu_1,\mu_2)\big)}, 1 - \frac{\max(\mu_1,\mu_2)}{2\big(1-\min(\mu_1,\mu_2)\big)}\right).$$

3. *If $\mu_1 + \mu_2 \geq 1$ and $\min(\mu_1,\mu_2) < 2\max(\mu_1,\mu_2)\big(1 - \max(\mu_1,\mu_2)\big)$, then*
$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 \geq \mu_2, \\ (1-\mu_2, \mu_2) & \text{if } \mu_1 < \mu_2, \end{cases}$$
$$\boldsymbol{w} = \left(\frac{\min(\mu_1,\mu_2)}{2\big(1-\max(\mu_1,\mu_2)\big)}, 1 - \frac{\min(\mu_1,\mu_2)}{2\big(1-\max(\mu_1,\mu_2)\big)}\right).$$

4. *If $\mu_1 + \mu_2 \geq 1$ and $\min(\mu_1,\mu_2) \geq 2\max(\mu_1,\mu_2)\big(1 - \max(\mu_1,\mu_2)\big)$, then*
$$\boldsymbol{p} = \begin{cases} \left(1 - \frac{\mu_2}{2\mu_1}, \frac{\mu_2}{2\mu_1}\right) & \text{if } \mu_1 \geq \mu_2, \\ \left(\frac{\mu_1}{2\mu_2}, 1 - \frac{\mu_1}{2\mu_2}\right) & \text{if } \mu_1 < \mu_2, \end{cases}$$
$$\boldsymbol{w} = \big(\max(\mu_1,\mu_2), 1 - \max(\mu_1,\mu_2)\big).$$

Then $\mu = v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}$; that is, $\mathscr{C}_\mu = S^{U_P}_{\boldsymbol{p},\boldsymbol{w}}$.

Proof. Let μ be a normalized capacity on $N = \{1,2\}$ and recall that, when $n = 2$, the semi-uninorm U_P is defined by

$$U_P(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [0.5,1]^2, \\ \min(x,y) & \text{if } (x,y) \in [0,0.5)^2, \\ 2xy & \text{otherwise.} \end{cases}$$

We distinguish the following cases:

1. If $\mu_1 + \mu_2 < 1$, and $\max(\mu_1,\mu_2) < 2\min(\mu_1,\mu_2)\big(1 - \min(\mu_1,\mu_2)\big)$, then notice that the case $\min(\mu_1,\mu_2) = 0$ is not possible. Moreover, $\min(\mu_1,\mu_2) < 0.5$, and we also have
$$\frac{\max(\mu_1,\mu_2)}{2\min(\mu_1,\mu_2)} < 1 - \min(\mu_1,\mu_2),$$

or, equivalently,

$$\min(\mu_1,\mu_2) < 1 - \frac{\max(\mu_1,\mu_2)}{2\min(\mu_1,\mu_2)}.$$

On the other hand, since $\min(\mu_1,\mu_2) \le \max(\mu_1,\mu_2)$, we get

$$\frac{\max(\mu_1,\mu_2)}{2\min(\mu_1,\mu_2)} \ge 0.5,$$

and, consequently,

$$1 - \frac{\max(\mu_1,\mu_2)}{2\min(\mu_1,\mu_2)} \le 0.5.$$

Consider now the following weighting vectors:

$$\boldsymbol{p} = \begin{cases} \left(1 - \frac{\mu_2}{2\mu_1}, \frac{\mu_2}{2\mu_1}\right) & \text{if } \mu_1 \le \mu_2, \\ \left(\frac{\mu_1}{2\mu_2}, 1 - \frac{\mu_1}{2\mu_2}\right) & \text{if } \mu_1 > \mu_2, \end{cases}$$
$$\boldsymbol{w} = \big(\min(\mu_1,\mu_2), 1 - \min(\mu_1,\mu_2)\big).$$

We distinguish three cases:

(a) If $\mu_1 < \mu_2$, then

$$U_P(p_1,w_1) = \min\left(1 - \frac{\mu_2}{2\mu_1}, \mu_1\right) = \mu_1,$$
$$U_P(p_2,w_1) = 2\frac{\mu_2}{2\mu_1}\mu_1 = \mu_2.$$

(b) If $\mu_1 = \mu_2$, then

$$U_P(p_1,w_1) = \min(0.5,\mu_1) = \mu_1,$$
$$U_P(p_2,w_1) = \min(0.5,\mu_2) = \mu_2.$$

(c) If $\mu_1 > \mu_2$, then

$$U_P(p_1,w_1) = 2\frac{\mu_1}{2\mu_2}\mu_2 = \mu_1,$$
$$U_P(p_2,w_1) = \min\left(1 - \frac{\mu_1}{2\mu_2}, \mu_2\right) = \mu_2.$$

2. If $\mu_1 + \mu_2 < 1$ and $\max(\mu_1,\mu_2) \ge 2\min(\mu_1,\mu_2)\big(1 - \min(\mu_1,\mu_2)\big)$, then notice that the case $\min(\mu_1,\mu_2) = 1$ is not possible. Moreover, we have

$$\frac{\max(\mu_1,\mu_2)}{2\big(1-\min(\mu_1,\mu_2)\big)} \ge \min(\mu_1,\mu_2).$$

On the other hand, given that $\max(\mu_1,\mu_2) < 1 - \min(\mu_1,\mu_2)$, we get

$$\frac{\max(\mu_1,\mu_2)}{2\big(1-\min(\mu_1,\mu_2)\big)} < 0.5,$$

and, consequently, $\min(\mu_1,\mu_2) < 0.5$. Now consider the following weighting vectors:

$$\boldsymbol{p} = \begin{cases} (\mu_1, 1-\mu_1) & \text{if } \mu_1 \le \mu_2, \\ (1-\mu_2, \mu_2) & \text{if } \mu_1 > \mu_2, \end{cases}$$
$$\boldsymbol{w} = \left(\frac{\max(\mu_1,\mu_2)}{2\big(1-\min(\mu_1,\mu_2)\big)}, 1 - \frac{\max(\mu_1,\mu_2)}{2\big(1-\min(\mu_1,\mu_2)\big)}\right).$$

We distinguish two cases:

(a) If $\mu_1 \le \mu_2$, then

$$U_P(p_1,w_1) = \min\left(\mu_1, \frac{\mu_2}{2(1-\mu_1)}\right) = \mu_1,$$
$$U_P(p_2,w_1) = 2(1-\mu_1)\frac{\mu_2}{2(1-\mu_1)} = \mu_2.$$

(b) If $\mu_1 > \mu_2$, then

$$U_P(p_1,w_1) = 2(1-\mu_2)\frac{\mu_1}{2(1-\mu_2)} = \mu_1,$$
$$U_P(p_2,w_1) = \min\left(\mu_2, \frac{\mu_1}{2(1-\mu_2)}\right) = \mu_2.$$

3. If $\mu_1 + \mu_2 \ge 1$ and $\min(\mu_1,\mu_2) < 2\max(\mu_1,\mu_2)\big(1 - \max(\mu_1,\mu_2)\big)$, then the proof of this case is similar to that of the second item in Theorem 4.

4. If $\mu_1 + \mu_2 \ge 1$ and $\min(\mu_1,\mu_2) \ge 2\max(\mu_1,\mu_2)\big(1 - \max(\mu_1,\mu_2)\big)$, then the proof of this case is similar to that of the third item in Theorem 4. □

5. Conclusion

SUOWA operators are a useful tool for dealing with situations where combining values by using both a weighted mean and a OWA type aggregation is necessary. Given that they are Choquet integral-based operators with respect to normalized capacities, they have some natural properties such as continuity, monotonicity, idempotency, compensativeness and homogeneity of degree 1. For this reason it seems interesting to analyze their behavior from different points of view. In this paper we have shown that in two dimensions, if we consider one of the following continuous semi-uninorms: $U_{T_{\mathbf{L}}}$, $U_{T_{\mathbf{M}}}$, $U_{\widetilde{P}}$ and U_P, then any Choquet integral with respect to a normalized capacity can be expressed as a SUOWA operator associated with the chosen semi-uninorm and two weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$, which are given explicitly.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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