SUOWA operators: Constructing semi-uninorms and analyzing specific cases

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Abstract

SUOWA operators are a new family of aggregation functions that simultaneously generalize weighted means and OWA operators. Semi-uninorms, which are an extension of uninorms by dispensing with the symmetry and associativity properties, play a fundamental role in their definition. In this paper we show several procedures to construct semi-uninorms. The first one allows us to obtain continuous semi-uninorms by using ordinal sums of aggregation operators while the second one is based on a combination of several given semi-uninorms. We also pay special attention to the smallest and the largest idempotent semi-uninorms and we point out some advantages of SUOWA operators over WOWA operators.

Keywords: Choquet integral, weighted means, OWA operators, semi-uninorms, SUOWA operators, WOWA operators.

1. Introduction

There exists a great variety of functions proposed in the literature for the task of aggregating information. Weighted means and ordered weighted averaging operators (OWA), introduced by Yager [1], are two of the best known. Although both families of functions are defined through weighting vectors, their behavior is completely different: in the case of weighted means, the values are weighted according to the reliability of the information sources, while in the case of OWA operators, the values are weighted in accordance with their relative position. The need of both weightings in fields as diverse as robotics, vision, fuzzy logic controllers, constraint satisfaction problems, scheduling, multicriteria aggregation problems and decision making has been reported by several authors (see, for instance, Torra [2, 3], Torra and Godo [4, pp. 160–161], Torra and Narukawa [5, pp. 150–151], Roy [6], Yager and Alajlan [7], Llamazares [8] and references therein). This fact has prompted the emergence of specific functions to deal with this class of problems.

A common approach in this context is to consider families of functions parametrized by two weighting vectors that generalize weighted means and OWA operators in the sense that one of these functions is obtained when the

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other one has a "neutral" behavior; that is, its weighting vector is that of the arithmetic mean (for an analysis of some of them, see Llamazares [9]). Two of the best-known solutions are the operator proposed by Engemann *et al.* [10], and the weighted OWA operator (WOWA) introduced by Torra [3]. The operator proposed by Engemann *et al.* [10], which has been rediscovered as MO2P (Roy [6]), IP-OWA operator (Merigó [11]), and HWAA operator (Lin and Jiang [12]), has some interesting properties but also has an important shortcoming: it is not monotonic (see Liu [13], Llamazares [9], Wang [14] and Lin [15]). For its part, WOWA operators have good properties given that they are Choquet integral-based operators with respect to normalized capacities (see Torra [16]). However, in some cases, their behavior is unsuitable to model certain problems (see Llamazares [9, 8]).

Semi-uninorm based ordered weighted averaging operators (SUOWA) were introduced by Llamazares [8] as an alternative to the previous functions. They are also defined as Choquet integral-based operators associated with normalized capacities that are defined in terms of two weight distributions (one for the weighted mean part, the other for the OWA operator part) and "assembled" by semi-uninorms. Since SUOWA operators are representable as Choquet integral with respect to normalized capacities, they share some good properties with WOWA operators. In addition, in the case of using idempotent semi-uninorms, they present some interesting additional properties (see Llamazares [8]).

Given that semi-uninorms play a fundamental role in the definition of SUOWA operators, in this paper we show two procedures to construct semi-uninorms. The first one, based on the notion of ordinal sums of aggregation operators, allows us to obtain continuous semi-uninorms. The second one, based in the combination of several semi-uninorms, allows us to construct new semi-uninorms that keep some properties of the former semi-uninorms. Especially interesting is the case where semi-uninorms are combined by means of weighted means. Under certain assumptions, the value of the SUOWA operator associated with the new semi-uninorm can be straightforward obtained combining with the same weighted mean the values of the SUOWA operators associated with the former semi-uninorms.

Another issue addressed in this paper is the analysis of the capacities and the SUOWA operators associated with the smallest and the largest idempotent semi-uninorms (which are, in fact, two well-known uninorms given by Yager and Rybalov [17]). We conclude the paper with some comparisons between SUOWA operators and WOWA operators.

The paper is organized as follows. In Section 2 we recall some basic properties of aggregation functions and the definition of the discrete Choquet integral. We also show that weighted means, OWA operators, WOWA operators and SUOWA operators are particular cases of this integral. Section 3 is devoted to the construction of semi-uninorms, which play a fundamental role in the definition of SUOWA operators. In Section 4 we analyze the SUOWA operators obtained from the smallest and the largest idempotent semi-uninorms. Finally, Section 5 is dedicated to show some advantages of SUOWA operators over WOWA operators.

2. Preliminaries on aggregation functions

Throughout the paper we will use the following notation: $N = \{1, \ldots, n\}$; given $A \subseteq N$, $|A|$ denotes the cardinality of A; vectors are denoted in bold; $\boldsymbol{\eta}$ denotes the tuple $(1/n, \ldots, 1/n) \in \mathbb{R}^n$. We write $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in N$. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, $[\cdot]$ and $(\cdot)$ denote permutations such that $x_{[1]} \geq \cdots \geq x_{[n]}$ and $x_{(1)} \leq \cdots \leq x_{(n)}$.

We now recall well known properties of aggregation functions.

Definition 1. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function.

1. F is symmetric if $F(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = F(x_1, \ldots, x_n)$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and for all permutation σ of N.
2. F is monotonic if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $F(\boldsymbol{x}) \geq F(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.
3. F is idempotent if $F(x, \ldots, x) = x$ for all $x \in \mathbb{R}$.
4. F is compensative (or internal) if $\min(\boldsymbol{x}) \leq F(\boldsymbol{x}) \leq \max(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
5. F is homogeneous of degree 1 (or ratio scale invariant) if $F(r\boldsymbol{x}) = rF(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and for all $r > 0$.

2.1. Choquet integral

Choquet integral-based operators have a wide variety of applications (see, for instance, Grabisch *et al.* [18] and Grabisch and Labreuche [19]). Choquet integral is based on the notion of capacity (see Choquet [20] and Murofushi and Sugeno [21]). The concept of capacity resembles that of probability measure but in the definition of the former additivity is replaced by monotonicity (see also fuzzy measures in Sugeno [22]). A game is then a generalization of a capacity where the monotonicity is no longer required.

Definition 2.

1. A game υ on N is a set function, $\upsilon : 2^N \longrightarrow \mathbb{R}$ satisfying $\upsilon(\varnothing) = 0$.
2. A capacity (or fuzzy measure) μ on N is a game on N satisfying $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$. In particular, it follows that $\mu : 2^N \longrightarrow [0, \infty)$. A capacity μ is said to be normalized if $\mu(N) = 1$.

A straightforward way to get a capacity from a game is to consider the monotonic cover of the game (see Maschler and Peleg [23] and Maschler *et al.* [24]).

Definition 3. Let υ be a game on N. The monotonic cover of υ is the set function $\hat{\upsilon}$ given by

$$\hat{\upsilon}(A) = \max_{B \subseteq A} \upsilon(B).$$

Some basic properties of $\hat{\upsilon}$ are given in the sequel.

Remark 1. Let υ be a game on N. Then:

1. $\hat{\upsilon}$ is a capacity.
2. If υ is a capacity, then $\hat{\upsilon} = \upsilon$.

3. If $\upsilon(A) \leq 1$ for all $A \subseteq N$ and $\upsilon(N) = 1$, then $\hat{\upsilon}$ is a normalized capacity.

Although the Choquet integral is usually defined as a functional (see, for instance, Choquet [20], Murofushi and Sugeno [21] and Denneberg [25]), in this paper we consider the Choquet integral as an aggregation function over $\mathbb{R}^n$ (see, for instance, Grabisch *et al.* [26, p. 181]). Moreover, we define the Choquet integral for all vectors of $\mathbb{R}^n$ instead of nonnegative vectors given that we are actually considering the asymmetric Choquet integral with respect to μ (on this, see again Grabisch *et al.* [26, p. 182]).

Definition 4. Let μ be a capacity on N. The Choquet integral with respect to μ is the function $C_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu(B_{(i)})(x_{(i)} - x_{(i-1)}),$$

where $B_{(i)} = \{(i), \ldots, (n)\}$, and we use the convention $x_{(0)} = 0$.

It is worth noting that Choquet integral-based operators possess desirable properties which are useful in certain information aggregation contexts (see, for instance, Grabisch *et al.* [26, p. 193 and p. 196]).

Remark 2. Let μ be a capacity on N. Then C_μ is continuous, monotonic and homogeneous of degree 1. Moreover, it is idempotent and compensative when μ is a normalized capacity.

Choquet integral can also be represented by using a decreasing sequences of values (see, for instance, Torra [16] and Llamazares [8]):

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu(A_{[i]})(x_{[i]} - x_{[i+1]}) \tag{1}$$

where $A_{[i]} = \{[1], \ldots, [i]\}$, and we use the convention $x_{[n+1]} = 0$.

From the previous expression, it is straightforward to express the Choquet integral as follows

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} (\mu(A_{[i]}) - \mu(A_{[i-1]}))x_{[i]}, \tag{2}$$

with the convention $A_{[0]} = \varnothing$, where the weights of the values $x_{[i]}$ are shown explicitly.

2.2. *Weighted means and OWA operators*

Weighted means and OWA operators (introduced by Yager [1]) are well-known functions in the theory of aggregation operators. Both classes of functions are defined in terms of weight distributions that add up to 1.

Definition 5. A vector $\boldsymbol{q} \in \mathbb{R}^n$ is a weighting vector if $\boldsymbol{q} \in [0, 1]^n$ and $\sum_{i=1}^{n} q_i = 1$.

Definition 6. Let $\boldsymbol{p}$ be a weighting vector. The weighted mean associated with $\boldsymbol{p}$ is the function $M_{\boldsymbol{p}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$M_{\boldsymbol{p}}(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i.$$

Definition 7. Let $\boldsymbol{w}$ be a weighting vector. The OWA operator associated with $\boldsymbol{w}$ is the function $O_{\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$O_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{[i]}.$$

It is well known that weighted means and OWA operators are a special type of Choquet integral (see, for instance, Fodor *et al.* [27], Grabisch [28, 29] or Llamazares [8]).

Remark 3.

1. If $\boldsymbol{p}$ is a weighting vector, then $M_{\boldsymbol{p}}$ is the Choquet integral with respect to the normalized capacity $\mu_{\boldsymbol{p}}(A) = \sum_{i \in A} p_i$.
2. If $\boldsymbol{w}$ is a weighting vector, then $O_{\boldsymbol{w}}$ is the Choquet integral with respect to the normalized capacity $\mu_{|\boldsymbol{w}|}(A) = \sum_{i=1}^{|A|} w_i$.

So, according to Remark 2, weighted means and OWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1. Moreover, in the case of OWA operators, given that the values of the variables are previously ordered in a decreasing way, they are also symmetric.

2.3. WOWA operators

WOWA operators were introduced by Torra [3] in order to consider situations where both the importance of information sources and the importance of values had to be taken into account. Initially they were defined by using monotonic functions that interpolates the points $\left(i/n, \sum_{j=1}^{i} w_j\right)$ together with the point $(0, 0)$. But, given that quantifiers satisfy these properties, Torra and Godo [30] suggested an alternative definition by using these functions. The relationship between quantifiers and the weighting vectors $\boldsymbol{w}$ was given by Yager [31].

Definition 8. A function $Q : [0, 1] \longrightarrow [0, 1]$ is a quantifier if it satisfies the following properties:

1. $Q(0) = 0$, $Q(1) = 1$.
2. $x > y \Rightarrow Q(x) \geq Q(y)$; i.e., it is a monotonic function.

Given a quantifier Q, we can obtain a weighting vector $\boldsymbol{w}$ by means of the following expression:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, \dots, n.$$

In this case, we will say that the quantifier Q generates the weighting vector $\boldsymbol{w}$. Notice that Q interpolates the points $\left(i/n, \sum_{j=1}^{i} w_j\right)$ together with the point $(0, 0)$.

Definition 9. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let Q be a quantifier generating the weighting vector $\boldsymbol{w}$. The WOWA operator associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and Q is the function $W^Q_{\boldsymbol{p},\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$W^Q_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} q_i x_{[i]},$$

where the weight q_i is defined as

$$q_i = Q\left(\sum_{j=1}^{i} p_{[j]}\right) - Q\left(\sum_{j=1}^{i-1} p_{[j]}\right).$$

It is worth noting that, in order to generalize the weighted mean $M_{\boldsymbol{p}}$, it is necessary that the quantifier Q be the identity when $\boldsymbol{w} = \boldsymbol{\eta}$. Likewise, notice that WOWA operators are a specific case of Choquet integral (see Torra [16]).

Remark 4. If $\boldsymbol{p}$ and $\boldsymbol{w}$ are two weighting vectors and Q is a quantifier generating the weighting vector $\boldsymbol{w}$, then $\mu(A) = Q\left(\sum_{i\in A} p_i\right)$ is a normalized capacity on N and $\mathcal{C}_\mu$ is the WOWA operator $W^Q_{\boldsymbol{p},\boldsymbol{w}}$. Moreover, according to (1), the WOWA operator can also be written as

$$W^Q_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} Q\left(\sum_{j=1}^{i} p_{[j]}\right)(x_{[i]} - x_{[i+1]}). \tag{3}$$

WOWA operators generalize weighted means and OWA operators in the sense that $W^Q_{\boldsymbol{p},\boldsymbol{\eta}} = M_{\boldsymbol{p}}$ and $W^Q_{\boldsymbol{\eta},\boldsymbol{w}} = O_{\boldsymbol{w}}$. Moreover, according to Remark 2, they are continuous, monotonic, idempotent, compensative and homogeneous of degree 1 (see Torra [3]).

2.4. SUOWA operators

SUOWA operators were introduced by Llamazares [8] as an alternative to WOWA operators. These functions are defined through Choquet integral where their capacities are constructed by using semi-uninorms and the values of the capacities associated with the weighted means and the OWA operators. Semi-uninorms, studied by Liu [32], are monotonic functions with a neutral element in the interval $[0,1]$. These functions were introduced as a generalization of uninorms, by dispensing with the symmetry and associativity properties. In turn, uninorms were proposed by Yager and Rybalov [17] as a generalization of t-norms and t-conorms (see also Fodor *et al.* [33], and Fodor and De Baets [34]).

Definition 10. Let $U : [0,1]^2 \longrightarrow [0,1]$.

1. U is a semi-uninorm if it is monotonic and possesses a neutral element $e \in [0,1]$ ($U(e,x) = U(x,e) = x$ for all $x \in [0,1]$).
2. U is a uninorm if it is a symmetric and associative ($U(x,U(y,z)) = U(U(x,y),z)$ for all $x,y,z \in [0,1]$) semi-uninorm.

We denote by $\mathcal{U}^e$ (respectively, $\mathcal{U}^e_{\mathrm{i}}$) the set of semi-uninorms (respectively, idempotent semi-uninorms) with neutral element $e \in [0,1]$. The structure of semi-uninorms and idempotent semi-uninorms has been studied by Liu [32, Propositions 2.1 and 2.2] and it is represented in Figure 1.

SUOWA operators are Choquet integral-based operators where their capacities are the monotonic cover of specific games. These games are defined by using semi-uninorms with neutral element $1/n$ and the values of the capacities

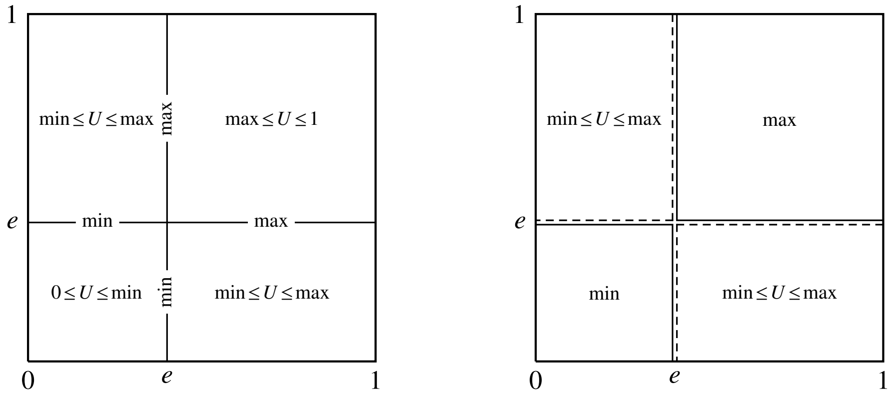

Figure 1: The structure of semi-uninorms and idempotent semi-uninorms, respectively.

associated with the weighted means and the OWA operators. To guarantee that the monotonic cover of the game is a normalized capacity, we restrict our attention to the following subset of semi-uninorms (see Llamazares [8]):

$$\widetilde{\mathcal{U}}^{1/n} = \left\{ U \in \mathcal{U}^{1/n} \mid U(1/k, 1/k) \leq 1/k \text{ for all } k \in N \right\}.$$

Obviously $\mathcal{U}_{\mathrm{i}}^{1/n} \subseteq \widetilde{\mathcal{U}}^{1/n}$. Moreover, it is easy to check that the smallest and the largest elements of $\widetilde{\mathcal{U}}^{1/n}$ are, respectively, the following semi-uninorms:

$$U_{\perp}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n, 1]^2, \\ 0 & \text{if } (x,y) \in [0, 1/n)^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

and

$$U_{\top}(x,y) = \begin{cases} 1/k & \text{if } (x,y) \in I_k \setminus I_{k+1}, \text{ where } I_k = (1/n, 1/k]^2 \quad (k \in N \setminus \{n\}), \\ \min(x,y) & \text{if } (x,y) \in [0, 1/n]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

In the case of idempotent semi-uninorms, the smallest and the largest elements of $\mathcal{U}_{\mathrm{i}}^{1/n}$ are, respectively, the following uninorms (which were given by Yager and Rybalov [17]):

$$U_{\min}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n, 1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

and

$$U_{\max}(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0, 1/n]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

The games from which SUOWA operators are built are defined as follows.

Definition 11. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let $U \in \widetilde{\mathcal{U}}^{1/n}$.

1. The game associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U is the set function $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U} : 2^N \longrightarrow \mathbb{R}$ defined by

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}(A) = |A|\, U\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, \frac{\mu_{|\boldsymbol{w}|}(A)}{|A|}\right)$$

   if $A \neq \varnothing$, and $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}(\varnothing) = 0$.

2. $\hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}$, the monotonic cover of the game $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}$, will be called the capacity associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U.

Notice that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}(A) \leq 1$ for all $A \subseteq N$ and $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}(N) = 1$. Therefore, according to the third item of Remark 1, $\hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}$ is always a normalized capacity.

Definition 12. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let $U \in \widetilde{\mathcal{U}}^{1/n}$. The SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U is the function $S_{\boldsymbol{p},\boldsymbol{w}}^{U} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$S_{\boldsymbol{p},\boldsymbol{w}}^{U}(\boldsymbol{x}) = \sum_{i=1}^{n} s_i x_{[i]},$$

where $s_i = \hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}(A_{[i]}) - \hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}(A_{[i-1]})$ for all $i \in N$, $\hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}$ is the capacity associated with $\boldsymbol{p}, \boldsymbol{w}$ and U, and $A_{[i]} = \{[1], \dots, [i]\}$ (with the convention that $A_{[0]} = \varnothing$).

According to (1), the SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U can also be written as

$$S_{\boldsymbol{p},\boldsymbol{w}}^{U}(\boldsymbol{x}) = \sum_{i=1}^{n} \hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}(A_{[i]})(x_{[i]} - x_{[i+1]}). \tag{4}$$

By the choice of $\hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}$ we have $S_{\boldsymbol{p},\boldsymbol{\eta}}^{U} = M_{\boldsymbol{p}}$ and $S_{\boldsymbol{\eta},\boldsymbol{w}}^{U} = O_{\boldsymbol{w}}$ for any $U \in \widetilde{\mathcal{U}}^{1/n}$. Moreover, by Remark 2 and given that $\hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}$ is a normalized capacity, SUOWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1.

3. Constructing semi-uninorms

A central issue in the field of SUOWA operators is the choice of the semi-uninorm used for constructing the game $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}$. So, this section is devoted to illustrate the construction of semi-uninorms. Firstly, we show how to get continuous semi-uninorms belonging to $\widetilde{\mathcal{U}}^{1/n}$ by using ordinal sums of aggregation operators. After that, we focus on how to combine several semi-uninorms belonging to $\widetilde{\mathcal{U}}^{1/n}$ to obtain another semi-uninorm of $\widetilde{\mathcal{U}}^{1/n}$.

3.1. *Continuous semi-uninorms*

Notice that the semi-uninorms $U_\bot$, $U_\top$, $U_{\min}$ and $U_{\max}$ can also be obtained as ordinal sums of aggregation operators (see De Baets and Mesiar [35]). By using also ordinal sums (adapting to our context the result shown by Calvo *et al.* [36, pp. 37–38] and Grabisch *et al.* [26, pp. 264–265]) we get continuous semi-uninorms that coincide with a given semi-uninorm at the subset $[0, 1/n]^2 \cup [1/n, 1]^2$ (obviously under the assumption that the restrictions of this semi-uninorm to the sets $[0, 1/n]^2$ and $[1/n, 1]^2$ are continuous). We will denote by $\widetilde{\mathcal{U}}_{\mathrm{c}}^{1/n}$ the set of continuous semi-uninorms belonging to $\widetilde{\mathcal{U}}^{1/n}$; that is,

$$\widetilde{\mathcal{U}}_{\mathrm{c}}^{1/n} = \left\{U \in \widetilde{\mathcal{U}}^{1/n} \mid U \text{ is continuous}\right\}.$$

Proposition 1. *Let* $h : [0, 1] \longrightarrow [-\infty, \infty]$ *be any continuous strictly monotonic function with* $\mathrm{ran}(h) \neq [-\infty, \infty]$*, let* $U \in \widetilde{\mathcal{U}}^{1/n}$ *be such that* $U\big|_{[0,1/n]^2}$ *and* $U\big|_{[1/n,1]^2}$ *are continuous, and let* $U_h : [0, 1]^2 \longrightarrow [0, 1]$ *be the mapping defined by*

$$U_h(x, y) = h^{-1}\Big(h(U(\min(x, 1/n), \min(y, 1/n))) + h(U(\max(x, 1/n), \max(y, 1/n))) - h(1/n)\Big).$$

Then we have $U_h\big|_{[0,1/n]^2\cup[1/n,1]^2} = U\big|_{[0,1/n]^2\cup[1/n,1]^2}$ *and* $U_h \in \widetilde{\mathcal{U}}_{\mathrm{c}}^{1/n}$.

Notice that U_h can also be represented as

$$U_h(x, y) = \begin{cases} U(x, y) & \text{if } (x, y) \in [0, 1/n]^2 \cup [1/n, 1]^2, \\ h^{-1}(h(x) + h(y) - h(1/n)) & \text{otherwise.} \end{cases}$$

Two important families of continuous semi-uninorms are obtained when we consider the functions $h(x) = x$ and $h(x) = \ln x$. In the first case, the semi-uninorm U_h is given by

$$\begin{aligned} U_h(x, y) &= U(\min(x, 1/n), \min(y, 1/n)) + U(\max(x, 1/n), \max(y, 1/n)) - 1/n \\ &= \begin{cases} U(x, y) & \text{if } (x, y) \in [0, 1/n]^2 \cup [1/n, 1]^2, \\ x + y - 1/n & \text{otherwise,} \end{cases} \end{aligned}$$

while in the second case, when $h(x) = \ln x$, the semi-uninorm U_h is defined as

$$\begin{aligned} U_h(x, y) &= n\, U(\min(x, 1/n), \min(y, 1/n))\, U(\max(x, 1/n), \max(y, 1/n)) \\ &= \begin{cases} U(x, y) & \text{if } (x, y) \in [0, 1/n]^2 \cup [1/n, 1]^2, \\ nxy & \text{otherwise.} \end{cases} \end{aligned}$$

There is also another interesting class of continuous semi-uninorms that falls into the family obtained when $h(x) = x$. If $R : [0, 1]^2 \longrightarrow [0, 1]$ is a quasi-copula (see Alsina *et al.* [37] and Genest *et al.* [38]), then the continuous semi-uninorm

$$U_R(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [1/n, 1]^2, \\ R(nx, ny)/n & \text{if } (x, y) \in [0, 1/n)^2, \\ x + y - 1/n & \text{otherwise,} \end{cases}$$

is a 1-Lipschitz aggregation operator with neutral element $1/n$ (see Kolesárová [39]). Notice that when we consider the largest quasi-copula, i.e. the minimum operator, we obtain the idempotent semi-uninorm

$$U_{T_{\mathbf{M}}}(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [1/n, 1]^2, \\ \min(x, y) & \text{if } (x, y) \in [0, 1/n)^2, \\ x + y - 1/n & \text{otherwise.} \end{cases}$$

On the other hand, when we consider the smallest quasi-copula, i.e. the Łukasiewicz t-norm, we get

$$U_{T_{\mathbf{L}}}(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [1/n, 1]^2, \\ \max(x + y - 1/n, 0) & \text{otherwise.} \end{cases}$$

Notice that in the semi-uninorm $U_{T_{\mathbf{L}}}$ the expression $x + y - 1/n$ is extended to the region $[0, 1/n]^2$ as much as possible. Based on this idea, in Proposition 1 we can consider $U \in \widetilde{\mathcal{U}}^{1/n}$ such that $U(x, y) = h^{(-1)}(h(x) + h(y) - h(1/n))$ for all $(x, y) \in [0, 1/n]^2$, where $h^{(-1)}$ is the pseudo-inverse of h (see Klement *et al.* [40, 41]). From the properties of $h^{(-1)}$, we have the following corollary.

Corollary 1. *Let $h : [0, 1] \longrightarrow [-\infty, \infty]$ be any continuous strictly monotone function with ran$(h) \neq [-\infty, \infty]$, let $U \in \widetilde{\mathcal{U}}^{1/n}$ be such that $U\big|_{[1/n,1]^2}$ is continuous, and let $U_{(h)} : [0, 1]^2 \longrightarrow [0, 1]$ be the mapping defined by*

$$U_{(h)}(x, y) = \begin{cases} U(x, y) & \textit{if } (x, y) \in [1/n, 1]^2, \\ h^{(-1)}(h(x) + h(y) - h(1/n)) & \textit{otherwise.} \end{cases}$$

Then $U_{(h)} \in \widetilde{\mathcal{U}}_{\mathrm{c}}^{1/n}$.

For instance, when $U\big|_{[1/n,1]^2} = \max$ and we consider the functions $h(x) = x$ and $h(x) = \ln x$, we get, respectively, $U_{T_{\mathbf{L}}}$ and

$$U_{(h)}(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [1/n, 1]^2, \\ nxy & \text{otherwise.} \end{cases}$$

This last semi-uninorm will be denoted by $U_{\widetilde{P}}$.

To finish with this subsection, notice that Proposition 1 also allows us to obtain continuous idempotent semi-uninorms. For any $U \in \mathcal{U}_{\mathrm{i}}^{1/n}$, we have that $U_h \in \mathcal{U}_{\mathrm{i}}^{1/n}$ and

$$\begin{aligned} U_h(x, y) &= h^{-1}\Big(h\big(\min(x, y, 1/n)\big) + h\big(\max(x, y, 1/n)\big) - h(1/n)\Big) \\ &= \begin{cases} \max(x, y) & \text{if } (x, y) \in [1/n, 1]^2, \\ \min(x, y) & \text{if } (x, y) \in [0, 1/n)^2, \\ h^{-1}(h(x) + h(y) - h(1/n)) & \text{otherwise.} \end{cases} \end{aligned}$$

The particular cases of $h(x) = x$ and $h(x) = \ln x$ allow us to get, respectively, the continuous idempotent semi-uninorms

$$U_h(x, y) = \min(x, y, 1/n) + \max(x, y, 1/n) - 1/n = U_{T_{\mathbf{M}}}(x, y)$$

and

$$U_h(x, y) = n\,\min(x, y, 1/n)\max(x, y, 1/n) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [1/n, 1]^2, \\ \min(x, y) & \text{if } (x, y) \in [0, 1/n)^2, \\ nxy & \text{otherwise.} \end{cases}$$

This last semi-uninorm will be denoted by U_P. The relationship between $U_{T_{\mathbf{M}}}$ and U_P is given in the following proposition.

Proposition 2. *For each $n \in \mathbb{N}$, $U_{T_{\mathbf{M}}} \geq U_P$.*

Proof. By definition of $U_{T_{\mathbf{M}}}$ and U_P, the equality between both semi-uninorms holds on $[0, 1/n]^2 \cup [1/n, 1]^2$. Consider now $(x, y) \in (1/n, 1] \times [0, 1/n)$. Then

$$x + y - \frac{1}{n} \geq nxy \;\Leftrightarrow\; x(1 - ny) + y \geq \frac{1}{n},$$

and the last inequality is derived as follows:

$$x(1 - ny) + y \geq \frac{1}{n}(1 - ny) + y = \frac{1}{n}.$$

The case $(x, y) \in [0, 1/n) \times (1/n, 1]$ can be proven in a similar way. □

3.2. Combining semi-uninorms

Besides the ordinal sums of aggregation operators, another way for constructing semi-uninorms belonging to the set $\widetilde{\mathcal{U}}^{1/n}$ is by means of monotonic and idempotent functions that combine elements of $\widetilde{\mathcal{U}}^{1/n}$.

Proposition 3. *Let $U_1, \ldots, U_m \in \widetilde{\mathcal{U}}^{1/n}$, let $f : [0, 1]^m \longrightarrow [0, 1]$ be monotonic and idempotent, and let $U = f(U_1, \ldots, U_m)$ (that is, $U(x, y) = f(U_1(x, y), \ldots, U_m(x, y))$ for all $(x, y) \in [0, 1]^2$). Then the following holds:*

1. *$U \in \widetilde{\mathcal{U}}^{1/n}$.*
2. *If $U_1, \ldots, U_m \in \mathcal{U}_{\mathrm{i}}^{1/n}$, then $U \in \mathcal{U}_{\mathrm{i}}^{1/n}$.*
3. *If $U_1, \ldots, U_m \in \widetilde{\mathcal{U}}_{\mathrm{c}}^{1/n}$ and f is continuous, then $U \in \widetilde{\mathcal{U}}_{\mathrm{c}}^{1/n}$.*
4. *If $U_1, \ldots, U_m$ are symmetric, then U is also symmetric.*

Proof. It is straightforward and therefore omitted. □

As we show in the following proposition, if we combine semi-uninorms through monotonic, idempotent and homogeneous of degree 1 functions, then the games associated with these new semi-uninorms can be straightforward obtained combining with the same functions the games associated with the former semi-uninorms. Moreover, the fact of being a normalized capacity are retained from the former semi-uninorms.

Proposition 4. *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, let $U_1, \dots, U_m \in \widetilde{\mathcal{U}}^{1/n}$, let $f : [0,1]^m \longrightarrow [0,1]$ be monotonic, idempotent and homogeneous of degree 1,*[1] *and let $U = f(U_1, \dots, U_m)$. Then:*

1. $\upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}(A) = f(\upsilon^{U_1}_{\boldsymbol{p},\boldsymbol{w}}(A), \dots, \upsilon^{U_m}_{\boldsymbol{p},\boldsymbol{w}}(A))$ *for any subset A of N.*
2. *If $\upsilon^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \dots, \upsilon^{U_m}_{\boldsymbol{p},\boldsymbol{w}}$ are normalized capacities, then $\upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}$ is also a normalized capacity.*

Proof. It is straightforward and therefore omitted. □

Notice that the property $\hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(A) = f(\hat{\upsilon}^{U_1}_{\boldsymbol{p},\boldsymbol{w}}(A), \dots, \hat{\upsilon}^{U_m}_{\boldsymbol{p},\boldsymbol{w}}(A))$ is not satisfied in general. For instance, consider Example 3 in Llamazares [8], where $\boldsymbol{p} = (0.6, 0.2, 0.1, 0.1)$, $\boldsymbol{w} = (0.4, 0, 0, 0.6)$, $m = 2$, $U_1 = U_{\min}$, $U_2 = U_{\max}$, f is the arithmetic mean and $A = \{1, 2\}$. Then,

$$\hat{\upsilon}^{U_{\text{am}}}_{\boldsymbol{p},\boldsymbol{w}}(\{1,2\}) = 0.6$$
$$f(\hat{\upsilon}^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(\{1,2\}), \hat{\upsilon}^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}(\{1,2\})) = f(0.6, 0.8) = 0.7.$$

Some typical examples of monotonic, idempotent and homogeneous of degree 1 functions are the following:

1. The weighted mean $f(\boldsymbol{x}) = \sum_{j=1}^{m} \lambda_j x_j$, where $\boldsymbol{\lambda}$ is a weighting vector.
2. The weighted geometric mean $f(\boldsymbol{x}) = \prod_{j=1}^{m} x_j^{\lambda_j}$, where $\boldsymbol{\lambda}$ is a weighting vector.
3. The weighted root-mean-power $f(\boldsymbol{x}) = \left(\sum_{j=1}^{m} \lambda_j x_j^{\alpha}\right)^{1/\alpha}$, where $\boldsymbol{\lambda}$ is a weighting vector and $\alpha \neq 0$.

In the case of combining semi-uninorms through weighted means, if the games associated with the semi-uninorms are normalized capacities, then the value of the SUOWA operator associated with the new semi-uninorm can be straightforward obtained combining with the same weighted mean the values of the SUOWA operators associated with the former semi-uninorms.

Proposition 5. *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, let $U_1, \dots, U_m \in \widetilde{\mathcal{U}}^{1/n}$ such that $\upsilon^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \dots, \upsilon^{U_m}_{\boldsymbol{p},\boldsymbol{w}}$ be normalized capacities, let $\boldsymbol{\lambda}$ be a weighting vector, and let $U = \sum_{j=1}^{m} \lambda_j U_j$. Then*

$$S^{U}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{j=1}^{m} \lambda_j S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{R}^n$.

Proof. According to expression (4) and Proposition 4, given $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$S^{U}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} \upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]})(x_{[i]} - x_{[i+1]}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_j \upsilon^{U_j}_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]})(x_{[i]} - x_{[i+1]}) = \sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} \upsilon^{U_j}_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]})(x_{[i]} - x_{[i+1]})$$
$$= \sum_{j=1}^{m} \lambda_j S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}).$$
□

[1] Since f is defined on $[0,1]^m$, we consider the following definition of homogeneity of degree 1 instead of (5) in Definition 1: $f(r\boldsymbol{x}) = rf(\boldsymbol{x})$ for all $r > 0$ and all $\boldsymbol{x} \in [0,1]^m$ such that $r\boldsymbol{x} \in [0,1]^m$.

4. The cases $U_{\min}$ and $U_{\max}$

Among the great variety of semi-uninorms belonging to $\widetilde{\mathcal{U}}^{1/n}$ that could be chosen to generate a SUOWA operator, idempotent semi-uninorms are of specific interest owing to their notable properties (see Llamazares [8]).

Proposition 6. *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, and $U \in \mathcal{U}_{\mathrm{i}}^{1/n}$. Then the following holds:*

1. *For any nonempty subset A of N, we have*

$$\min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) \leq \upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}(A) \leq \hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(A) \leq \max\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right).$$

2. *Let $\boldsymbol{x} \in \mathbb{R}^n$ such that $p_{[i]} = w_i$ for all $i \in N$. Then $s_i = p_{[i]} = w_i$ for all $i \in N$ and, consequently,*

$$S^{U}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = M_{\boldsymbol{p}}(\boldsymbol{x}) = O_{\boldsymbol{w}}(\boldsymbol{x}).$$

3. *For any $\boldsymbol{x} \in \mathbb{R}^n$ we have*

$$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) \leq S^{U}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) \leq S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}).$$

The third item of the previous proposition shows that the SUOWA operators $S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$ and $S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$ are the bounds when we consider idempotent semi-uninorms. For this reason, in the next subsections we will focus on $U_{\min}$ and $U_{\max}$, and the corresponding SUOWA operators. Our aim is to show some interesting cases where the game associated with some weighting vectors and these uninorms is a normalized capacity.

4.1. The uninorm $U_{\min}$

The uninorm $U_{\min}$ can also be expressed as

$$U_{\min}(x,y) = \begin{cases} \min(x,y) & \text{if } \min(x,y) < 1/n, \\ \max(x,y) & \text{otherwise,} \end{cases}$$

and, as we show in the following remark, the game associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and $U_{\min}$ can be represented through this expression.

Remark 5. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors. Then, for any nonempty subset A of N, we get

$$\upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A) = |A|\, U_{\min}\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, \frac{\mu_{|\boldsymbol{w}|}(A)}{|A|}\right) = \begin{cases} \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) & \text{if } \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) < \dfrac{|A|}{n}, \\ \max\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) & \text{otherwise.} \end{cases}$$

Since

$$\hat{\upsilon}^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A) = \max_{B\subseteq A} \upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(B),$$

we have that $\hat{\upsilon}^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A) > \upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A)$ if and only if there exists $B \subsetneq A$ such that $\upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(B) > \upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A)$. For this, it is necessary and sufficient that the following conditions be satisfied:

$$\upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A) = \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right), \quad \upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(B) = \max\left(\sum_{i\in B} p_i, \sum_{i=1}^{|B|} w_i\right), \quad \max\left(\sum_{i\in B} p_i, \sum_{i=1}^{|B|} w_i\right) > \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right).$$

This fact is stated in the following proposition.

Proposition 7. *Let* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *be two weighting vectors. Given a nonempty subset A of N,* $\hat{\upsilon}^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A) > \upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A)$ *if and only if*

$$\min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) < \frac{|A|}{n}$$

and there exists a nonempty $B \subsetneq A$ *such that*

$$\min\left(\sum_{i\in B} p_i, \sum_{i=1}^{|B|} w_i\right) \geq \frac{|B|}{n} \quad \text{and} \quad \max\left(\sum_{i\in B} p_i, \sum_{i=1}^{|B|} w_i\right) > \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right).$$

In the sequel we give a condition on the weighting vector $\boldsymbol{w}$ that ensures that $\upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity for all weighting vectors $\boldsymbol{p}$. When this condition is satisfied, we also prove that the value returned by the SUOWA operator is less than or equal to the values returned by the weighted mean and the OWA operator. In addition, if the capacity associated with the OWA operator is less than or equal to the capacity associated with the weighted mean, then the SUOWA operator coincides with the OWA operator.

Proposition 8. *Let* $\boldsymbol{w}$ *be a weighting vector such that* $\sum_{i=1}^{j} w_i < j/n$ *for all* $j \in \{1,\dots,n-1\}$*. Then, for all weighting vector* $\boldsymbol{p}$*, we have:*

1. $\upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$ *is a normalized capacity on N.*
2. *For any nonempty subset A of N,*

$$\upsilon^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A) = \min\left(\mu_{\boldsymbol{p}}(A), \mu_{|\boldsymbol{w}|}(A)\right) = \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right).$$

3. *For all* $\boldsymbol{x} \in \mathbb{R}^n$,

$$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} s_i x_{[i]},$$

where, for all $i \in N$,

$$s_i = \min\left(\sum_{j=1}^{i} p_{[j]}, \sum_{j=1}^{i} w_j\right) - \min\left(\sum_{j=1}^{i-1} p_{[j]}, \sum_{j=1}^{i-1} w_j\right).$$

4. *For all* $\boldsymbol{x} \in \mathbb{R}^n$,

$$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) \leq \min\left(M_{\boldsymbol{p}}(\boldsymbol{x}), O_{\boldsymbol{w}}(\boldsymbol{x})\right).$$

5. *If* $\mu_{|\boldsymbol{w}|}(A) \leq \mu_{\boldsymbol{p}}(A)$ *for all* $A \subseteq N$*, then* $S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}} = O_{\boldsymbol{w}}$.

Proof. Since $\sum_{i=1}^{j} w_i < j/n$ for all $j \in \{1, \dots, n-1\}$, given a nonempty $A \subsetneq N$ we always have that

$$\min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) < \frac{|A|}{n}.$$

Therefore,

1. It is obvious by Proposition 7.
2. It is obvious by Remark 5.
3. It is obvious from the definition of SUOWA operator and the second item.
4. It is obvious from expression (1), Remark 3 and the second item.
5. It is obvious from the second item. □

From this result we can guarantee that, irrespective of the weighting vector $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$ is a normalized capacity on N when $\boldsymbol{w} = (w_1, \dots, w_n)$ is an increasing sequence of weights. Before that, we establish the following lemma.

Lemma 1. *Let $\boldsymbol{w}$ be a weighting vector such that $w_1 \leq w_2 \leq \cdots \leq w_n$. Then $w_1 = \cdots = w_n = 1/n$ or $\sum_{i=1}^{j} w_i < j/n$ for all $j \in \{1, \dots, n-1\}$.*

Proof. Let $\boldsymbol{w}$ be a weighting vector such that $w_1 \leq w_2 \leq \cdots \leq w_n$. We distinguish two cases:

1. If $w_1 = 1/n$, then, since $\sum_{i=1}^{n} w_i = 1$, we have $w_2 = \cdots = w_n = 1/n$.
2. If $w_1 < 1/n$, we are going to prove that $\sum_{i=1}^{j} w_i < j/n$ for all $j \in \{2, \dots, n-1\}$. This is proven by contradiction. Suppose that there exists $k \in \{2, \dots, n-1\}$ such that $\sum_{i=1}^{k} w_i \geq k/n$. In this case, we have $w_k > 1/n$. Moreover, since $\sum_{i=1}^{n} w_i = 1$, we have
$$1 = \sum_{i=1}^{k} w_i + \sum_{i=k+1}^{n} w_i \geq \frac{k}{n} + \sum_{i=k+1}^{n} w_i.$$
Therefore $\sum_{i=k+1}^{n} w_i \leq (n-k)/n$ and, consequently, $w_{k+1} \leq 1/n$, which clearly contradicts that $w_k > 1/n$. □

Corollary 2. *Let $\boldsymbol{w}$ be a weighting vector such that $w_1 \leq w_2 \leq \cdots \leq w_n$. Then, for all weighting vector $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$ is a normalized capacity on N, and $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}} = M_{\boldsymbol{p}}$ or $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}(\boldsymbol{x}) \leq \min\left(M_{\boldsymbol{p}}(\boldsymbol{x}), O_{\boldsymbol{w}}(\boldsymbol{x})\right)$ for all $\boldsymbol{x} \in \mathbb{R}^n$.*

Proof. Let $\boldsymbol{p}$ be a weighting vector. By Lemma 1 we distinguish two cases:

1. If $w_1 = \cdots = w_n = 1/n$, then $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}(A) = \upsilon_{\boldsymbol{p},\boldsymbol{\eta}}^{U_{\min}}(A) = \mu_{\boldsymbol{p}}(A)$ for all $A \subseteq N$. Therefore, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$ is a normalized capacity on N and $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}} = M_{\boldsymbol{p}}$.
2. If $\sum_{i=1}^{j} w_i < j/n$ for all $j \in \{1, \dots, n-1\}$, then, by Proposition 8, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$ is a normalized capacity on N and $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}(\boldsymbol{x}) \leq \min\left(M_{\boldsymbol{p}}(\boldsymbol{x}), O_{\boldsymbol{w}}(\boldsymbol{x})\right)$ for all $\boldsymbol{x} \in \mathbb{R}^n$. □

4.2. The uninorm $U_{\max}$

The analysis for the uninorm $U_{\max}$ is similar to that of the uninorm $U_{\min}$; so, the proofs are omitted. The uninorm $U_{\max}$ can also be expressed as

$$U_{\max}(x,y) = \begin{cases} \max(x,y) & \text{if } \max(x,y) > 1/n, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

and, consequently, the game associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and $U_{\max}$ can be represented by means of the following expression.

Remark 6. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors. Then, for any nonempty subset A of N, we have

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}(A) = |A|\, U_{\max}\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, \frac{\mu_{|\boldsymbol{w}|}(A)}{|A|}\right) = \begin{cases} \max\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) & \text{if } \max\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) > \frac{|A|}{n}, \\ \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) & \text{otherwise.} \end{cases}$$

We now characterize the nonempty subsets A of N for which the game and the capacity associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and $U_{\max}$ take different values.

Proposition 9. *Let* $\mathbf{p}$ *and* $\mathbf{w}$ *be two weighting vectors. Given a nonempty subset* A *of* N, $\hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}(A) > \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}(A)$ *if and only if*

$$\max\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) \le \frac{|A|}{n}$$

and there exists a nonempty $B \subsetneq A$ *such that*

$$\max\left(\sum_{i\in B} p_i, \sum_{i=1}^{|B|} w_i\right) > \frac{|B|}{n} \quad \text{and} \quad \max\left(\sum_{i\in B} p_i, \sum_{i=1}^{|B|} w_i\right) > \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right).$$

Next we give a condition on the weighting vector $\boldsymbol{w}$ that ensures that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$ is a normalized capacity for all weighting vectors $\boldsymbol{p}$. When this condition is satisfied, the value returned by the SUOWA operator is greater than or equal to the values returned by the weighted mean and the OWA operator. Moreover, if the capacity associated with the OWA operator is greater than or equal to the capacity associated with the weighted mean, then the SUOWA operator coincides with the OWA operator.

Proposition 10. *Let* $\mathbf{w}$ *be a weighting vector such that* $\sum_{i=1}^{j} w_i > j/n$ *for all* $j \in \{1,\dots,n-1\}$. *Then, for all weighting vector* $\boldsymbol{p}$, *we have:*

1. $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$ *is a normalized capacity on* N.
2. *For any nonempty subset* A *of* N,

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}(A) = \max\left(\mu_{\boldsymbol{p}}(A), \mu_{|\boldsymbol{w}|}(A)\right) = \max\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right).$$

3. *For all* $\boldsymbol{x} \in \mathbb{R}^n$,

$$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}(\boldsymbol{x}) = \sum_{i=1}^{n} s_i x_{[i]},$$

where, for all $i \in N$,

$$s_i = \max\left(\sum_{j=1}^{i} p_{[j]}, \sum_{j=1}^{i} w_j\right) - \max\left(\sum_{j=1}^{i-1} p_{[j]}, \sum_{j=1}^{i-1} w_j\right).$$

4. *For all* $\boldsymbol{x} \in \mathbb{R}^n$,

$$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}(\boldsymbol{x}) \geq \max\left(M_{\boldsymbol{p}}(\boldsymbol{x}), O_{\boldsymbol{w}}(\boldsymbol{x})\right).$$

5. *If* $\mu_{|\boldsymbol{w}|}(A) \geq \mu_{\boldsymbol{p}}(A)$ *for all* $A \subseteq N$, *then* $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}} = O_{\boldsymbol{w}}$.

As a consequence of this result we obtain that, irrespective of the weighting vector $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$ is a normalized capacity on N when $\boldsymbol{w} = (w_1, \ldots, w_n)$ is a decreasing sequence of weights. Before that, we establish the following lemma.

Lemma 2. *Let* $\boldsymbol{w}$ *be a weighting vector such that* $w_1 \geq w_2 \geq \cdots \geq w_n$. *Then* $w_1 = \cdots = w_n = 1/n$ *or* $\sum_{i=1}^{j} w_i > j/n$ *for all* $j \in \{1, \ldots, n-1\}$.

Corollary 3. *Let* $\boldsymbol{w}$ *be a weighting vector such that* $w_1 \geq w_2 \geq \cdots \geq w_n$. *Then, for all weighting vector* $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$ *is a normalized capacity on N, and* $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}} = M_{\boldsymbol{p}}$ *or* $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}(\boldsymbol{x}) \geq \max\left(M_{\boldsymbol{p}}(\boldsymbol{x}), O_{\boldsymbol{w}}(\boldsymbol{x})\right)$ *for all* $\boldsymbol{x} \in \mathbb{R}^n$.

5. Discussion

SUOWA and WOWA operators are obtained from Choquet integral with respect to normalized capacities. Therefore, both classes of operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1. Although they share many properties, they are different classes of aggregation operators as it has been pointed out by Llamazares [8]. In the case of WOWA operators, there is not a wide variety of interpolations methods for obtaining the quantifier Q, and they are relatively complex except in the case of linear interpolation (see Torra and Lv [42]). However, as we have seen in Section 3, there exits a wide variety of semi-uninorms by means of which we can generate a SUOWA operator associated with two weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$. This fact allows us a great flexibility when it comes to choosing a SUOWA operator, as we show in the following example.

Example 1. Let us consider the situation described by Torra and Godo [4, p. 160], where a robot is equipped with four sensors to determine the distance to the nearest object in the direction of its movement. Since the robot should give more importance to nearest objects than those that are further, a possible set of weights would be $\boldsymbol{w} = (0.1, 0.2, 0.3, 0.4)$. Suppose also that the sensors are of different quality and precision. This fact is easily modeled by using a weighting vector $\boldsymbol{p}$. In this example we take $\boldsymbol{p} = (0.3, 0.3, 0.2, 0.2)$.

Due to their good properties, we are going to focus on idempotent semi-uninorms. In the case of $U_{\min}$, Corollary 2 guarantees that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$ is a normalized capacity. As we can see in Table 1, this is also the case when we consider the semi-uninorms U_P, $U_{T_{\mathbf{M}}}$ and $U_{\max}$.

Table 1: Capacities associated with $U_{\min}$, U_P, $U_{T_\mathbf{M}}$ and $U_{\max}$

Set	$\hat{\upsilon}^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$\hat{\upsilon}^{U_P}_{\boldsymbol{p},\boldsymbol{w}}$	$\hat{\upsilon}^{U_{T_\mathbf{M}}}_{\boldsymbol{p},\boldsymbol{w}}$	$\hat{\upsilon}^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$
$\{1\}$	0.1	0.12	0.15	0.3
$\{2\}$	0.1	0.12	0.15	0.3
$\{3\}$	0.1	0.1	0.1	0.1
$\{4\}$	0.1	0.1	0.1	0.1
$\{1,2\}$	0.3	0.36	0.4	0.6
$\{1,3\}$	0.3	0.3	0.3	0.3
$\{1,4\}$	0.3	0.3	0.3	0.3
$\{2,3\}$	0.3	0.3	0.3	0.3
$\{2,4\}$	0.3	0.3	0.3	0.3
$\{3,4\}$	0.3	0.3	0.3	0.3
$\{1,2,3\}$	0.6	0.64	0.65	0.8
$\{1,2,4\}$	0.6	0.64	0.65	0.8
$\{1,3,4\}$	0.6	0.6	0.6	0.6
$\{2,3,4\}$	0.6	0.6	0.6	0.6
N	1	1	1	1

Consider now $\boldsymbol{x} = (9, 10, 5, 7)$. By expression (4), for any idempotent semi-uninorm U we have

$$S^{U}_{\boldsymbol{p},\boldsymbol{w}}(9, 10, 5, 7) = \hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(\{2\}) \cdot 1 + \hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(\{1, 2\}) \cdot 2 + \hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(\{1, 2, 4\}) \cdot 2 + \hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(N) \cdot 5.$$

So, the values obtained by using $U_{\min}$, U_P, $U_{T_\mathbf{M}}$ and $U_{\max}$ are

$$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(9, 10, 5, 7) = 6.9, \quad S^{U_P}_{\boldsymbol{p},\boldsymbol{w}}(9, 10, 5, 7) = 7.12, \quad S^{U_{T_\mathbf{M}}}_{\boldsymbol{p},\boldsymbol{w}}(9, 10, 5, 7) = 7.25, \quad S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}(9, 10, 5, 7) = 8.1.$$

Notice that, by the fifth item of Proposition 8, we have $S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(9, 10, 5, 7) = O_{\boldsymbol{w}}(9, 10, 5, 7)$. Moreover, in this example also happens $S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}(9, 10, 5, 7) = M_{\boldsymbol{p}}(9, 10, 5, 7)$. Likewise, by the third item of Proposition 6, for any idempotent semi-uninorm U we get

$$6.9 \le S^{U}_{\boldsymbol{p},\boldsymbol{w}}(9, 10, 5, 7) \le 8.1.$$

Notice that, by Proposition 5, we can easily obtain an idempotent semi-uninorms that allows us to get a SUOWA operator which takes in $(9, 10, 5, 7)$ a specific value in the range 6.9 to 8.1. For instance, if we look for an idempotent semi-uninorm U such that $S^{U}_{\boldsymbol{p},\boldsymbol{w}}(9, 10, 5, 7) = 7.8$, then, since $7.8 = 0.25 \cdot 6.9 + 0.75 \cdot 8.1$, it is sufficient to consider $U = 0.25\, U_{\min} + 0.75\, U_{\max}$.

Llamazares [8] also shows an example where SUOWA operators return a value which seems more consistent (regarding the values given by the weighted mean and the OWA operator) than the value returned by WOWA operators. In addition to the remarks made by this author, in the sequel we show other two differences between the behavior of SUOWA operators and WOWA operators. The first one is the following. Given $\boldsymbol{x} \in \mathbb{R}^n$, each value $x_{[i]}$ is associated with the weights $p_{[i]}$ and w_i. As we have seen in the second item of Proposition 6, if both weights are equal for all $i \in N$ and we consider idempotent semi-uninorms, then the corresponding weights of the SUOWA operator coincide with them and, consequently, the value provided by the SUOWA operator is the same as the value returned by the weighted mean and the OWA operator. This property is not satisfied in the case of WOWA operators, as we illustrate in the following example.

Example 2. Let us consider the weighting vectors $\boldsymbol{p} = \boldsymbol{w} = (0.4, 0.2, 0.2, 0.1, 0.1)$ and $\boldsymbol{x} = (10, 9, 7, 4, 4)$. It is easy to check that $M_{\boldsymbol{p}}(10, 9, 7, 4, 4) = O_{\boldsymbol{w}}(10, 9, 7, 4, 4) = 8$ and, according to the second item of Proposition 6, $S^{U}_{\boldsymbol{p},\boldsymbol{w}}(10, 9, 7, 4, 4) = 8$ for all $U \in \mathcal{U}^{1/n}_{\mathrm{i}}$.

Now, let Q be a quantifier generating the weighting vector $\boldsymbol{w}$; that is, a monotonic function that interpolates the points $(0, 0)$, $(0.2, 0.4)$, $(0.4, 0.6)$, $(0.6, 0.8)$, $(0.8, 0.9)$ and $(1, 1)$. The weights of the corresponding WOWA operator are

$$\begin{aligned}
q_1 &= Q(0.4) - Q(0) &&= 0.6,\\
q_2 &= Q(0.6) - Q(0.4) &&= 0.2,\\
q_3 &= Q(0.8) - Q(0.6) &&= 0.1,\\
q_4 &= Q(0.9) - Q(0.8) &&= Q(0.9) - 0.9,\\
q_5 &= Q(1) \quad - Q(0.9) &&= 1 - Q(0.9),
\end{aligned}$$

and $W^{Q}_{\boldsymbol{p},\boldsymbol{w}}(10, 9, 7, 4, 4) = 8.9$. So, whatever the quantifier used, $q_1 \neq p_{[1]} = w_1$, $q_3 \neq p_{[3]} = w_3$, and the value returned by the WOWA operator does not coincide with the value provided by the weighted mean and the OWA operator.

The second difference between SUOWA operators and WOWA operators is also related to the weights $p_{[i]}$ and w_i associated with $x_{[i]}$. Under certain conditions, SUOWA operators have a symmetrical behavior between the weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$.

Proposition 11. *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, U a symmetrical semi-uninorm belonging to $\widetilde{\mathcal{U}}^{1/n}$ and $\boldsymbol{x} \in \mathbb{R}^n$. If $\boldsymbol{p}'$ and $\boldsymbol{w}'$ are two weighting vectors such that $p'_{[i]} = w_i$ and $w'_i = p_{[i]}$ for all $i \in N$, and $\upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}$ and $\upsilon^{U}_{\boldsymbol{p}',\boldsymbol{w}'}$ are normalized capacities, then $S^{U}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = S^{U}_{\boldsymbol{p}',\boldsymbol{w}'}(\boldsymbol{x})$.*

Proof. According to (4), it is sufficient to prove that $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]}) = \upsilon^U_{\boldsymbol{p}',\boldsymbol{w}'}(A_{[i]})$ for all $i \in N$. Given $i \in N$,

$$\upsilon^U_{\boldsymbol{p}',\boldsymbol{w}'}(A_{[i]}) = i\,U\left(\frac{1}{i}\sum_{j=1}^{i} p'_{[j]}, \frac{1}{i}\sum_{j=1}^{i} w'_j\right) = i\,U\left(\frac{1}{i}\sum_{j=1}^{i} w_j, \frac{1}{i}\sum_{j=1}^{i} p_{[j]}\right) = i\,U\left(\frac{1}{i}\sum_{j=1}^{i} p_{[j]}, \frac{1}{i}\sum_{j=1}^{i} w_j\right) = \upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]}). \qquad \square$$

The previous property is not preserved by WOWA operators, as we show in the following example.

Example 3. Consider the weighting vectors of Example 2, $\boldsymbol{p} = \boldsymbol{w} = (0.4, 0.2, 0.2, 0.1, 0.1)$, and $\boldsymbol{x} = (7, 9, 10, 4, 4)$. If Q is a quantifier generating the weighting vector $\boldsymbol{w}$, then Q interpolates the points $(0, 0)$, $(0.2, 0.4)$, $(0.4, 0.6)$, $(0.6, 0.8)$, $(0.8, 0.9)$ and $(1, 1)$. According to expression (3) we get

$$W^Q_{\boldsymbol{p},\boldsymbol{w}}(7, 9, 10, 4, 4) = Q(0.2) \cdot 1 + Q(0.4) \cdot 2 + Q(8) \cdot 3 + Q(1) \cdot 4 = 8.3.$$

Let us consider now the weighting vectors $\boldsymbol{p}' = \boldsymbol{w}' = (0.2, 0.2, 0.4, 0.1, 0.1)$. Notice that $p'_{[i]} = w_i$ and $w'_i = p_{[i]}$ for all $i \in \{1, \dots, 5\}$. If Q' is a quantifier generating the weighting vector $\boldsymbol{w}'$, then Q' interpolates the points $(0, 0)$, $(0.2, 0.2)$, $(0.4, 0.4)$, $(0.6, 0.8)$, $(0.8, 0.9)$ and $(1, 1)$. According to expression (3) we get

$$W^{Q'}_{\boldsymbol{p}',\boldsymbol{w}'}(7, 9, 10, 4, 4) = Q'(0.4) \cdot 1 + Q'(0.6) \cdot 2 + Q'(8) \cdot 3 + Q'(1) \cdot 4 = 8.7.$$

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