SUOWA operators: An analysis of their conjunctive/disjunctive character

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Abstract

Choquet integrals (mainly weighted means and OWA operators) are a well-known family of functions widely used in several scientific fields. Because of this, it is very important to know the behavior of these functions in order to choose the best-suited operators in practical applications. In this sense, the analysis of the conjunctive/disjunctive character is very interesting given that allow understand the behavior of functions in relation to order statistics. In this paper we focus on some classes of SUOWA operators, which are Choquet integrals that simultaneously generalize weighted means and OWA operators, and provide conditions under which the k-conjunctive or k-disjunctive character of an OWA operator is retained by the SUOWA operators associated with it. Of particular interest is the case where the OWA operator is located between two order statistics, given that we can obtain SUOWA operators also located between the same order statistics. Likewise, we show closed-form expressions of k-conjunctiveness and k-disjunctiveness indices for some specific cases of SUOWA operators.

Keywords: Choquet integral, SUOWA operators, weighted means, OWA operators, order statistics, k-conjunctive and k-disjunctive functions.

1. Introduction

In the last years, the application of Choquet integral in several scientific fields has taken a growing interest (see, for instance, Grabisch [1, 2], Grabisch and Roubens [3], Grabisch and Labreuche [4, 5], Yager [6], and the references therein). This fact is due mainly to the Choquet integral has good properties for aggregation, is very versatile and allows taking into account the interaction that sometimes exists among the information sources (see, for instance, the classical example proposed by Grabisch [1, 2]).

Two important special cases of Choquet integrals are the weighted means and the ordered weighted averaging (OWA) operators (Yager [7]). Both families of functions have been frequently used as aggregation operators for evaluating alternatives (see, for instance, Torra [8]). It is worth noting that, although OWA operators are relatively recent, some classical decision criteria, such as Laplace, Wald, and Hurwicz criteria, are specific cases of them (see Wald [9], Hurwicz [10], and Milnor [11], and more recently Puerto et al. [12], and Ahn and Yager [13]).

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Weighted means and OWA operators are both defined by means of weighting vectors. In the first case, the weights allow to weigh each information source in relation to their importance whereas in the second case the weights weigh the values in accordance with their relative position. In a sense, it can be considered that both weightings are complementary, and several authors have pointed out the need for both of them in some fields like robotics, fuzzy logic controllers, constraint satisfaction problems, decision making and multicriteria aggregation problems (see, for instance, Torra and Godo [14, pp. 160–161], Torra and Narukawa [15, pp. 150–151], Roy [16], Yager and Alajlan [17], Llamazares [18] and the references therein). For this reason, several families of functions have been proposed in the literature to deal with this kind of problems.

A common approach is to consider families of functions parametrized by two weighting vectors, one for the weighted mean and the other one for the OWA type aggregation, that generalize weighted means and OWA operators in the sense that the weighted mean (or the OWA operator) is recovered when the other weighting vector is $(1/n, \ldots, 1/n)$ (see Llamazares [19] for an analysis of some of these families of functions). Two of the best-suited solutions are the weighted OWA operator (WOWA), introduced by Torra [20], and the semi-uninorm based ordered weighted averaging (SUOWA) operators, proposed by Llamazares [18], due basically to WOWA and SUOWA operators are specific cases of Choquet integral with respect to normalized capacities. So, they are continuous, monotonic, idempotent, compensative and homogeneous of degree 1 functions.

Besides the above properties, and in order to choose the best-suited operators in practical applications, it is also very relevant to understand the behavior of these functions with respect to other characteristics. For this purpose, several indices have been suggested in the literature: orness and andness degrees, Shapley value, interaction indices, veto and favor indices, k-conjunctiveness and k-disjunctiveness indices, etc. (see, for instance, Grabisch et al. [21, Chapter 10]). However, the main shortcoming of this approach is that closed-form expressions of some of these indices are only know for few operators (basically, weighted means and OWA operators).

In the case of SUOWA operators, a first study of some of these indices has been carried out by Llamazares [22]. In that paper, closed-form expressions of orness degree, Shapley value, and veto and favor indices were given for some special cases of SUOWA operators. One of the aims of this paper is to continue this line of research and provide closed-form expressions of k-conjunctiveness and k-disjunctiveness indices. These indices were suggested by Marichal [23] when it comes to studying the k-conjunctive and k-disjunctive character of Choquet integrals: A function is k-conjunctive if it is bounded from above by its kth lowest input value; i.e., by the kth order statistic, whereas it is k-disjunctive when is bounded from below by its kth highest input value; i.e., by the $(n - k + 1)$th order statistic. In this sense, in this paper we also provide conditions under which the k-conjunctive or k-disjunctive character of an OWA operator is retained by the SUOWA operators associated with it. Of particular interest is the case where the OWA operator is located between two order statistics, given that we can obtain SUOWA operators also located between the same order statistics. In this way, in the aggregation process is possible to rule out extreme values at the same time that the weights of the information sources are taken into account. So, these functions may be especially useful in some multi-attribute decision making problems where the presence of outliers in the data is detected.

The remainder of the paper is organized as follows. In Section 2 we present some basic properties in the context of aggregation and the notions of semi-uninorms and uninorms. Section 3 is devoted to Choquet integral, including the special cases of weighted means, OWA operators, and SUOWA operators. Likewise, the concepts of k-conjunctive and k-disjunctive functions, and the corresponding indices, are recalled. In Section 4 we give the main results of the paper. In Section 5 we show the applicability of SUOWA operators through an example. Finally, some concluding remarks are provided in Section 6. All proofs are in the Appendix.

2. Preliminaries

The following notation will be used throughout the paper: $N = \{1, \dots, n\}$; given $A \subseteq N$, $|A|$ denotes the cardinality of A; vectors are denoted in bold; $\boldsymbol{\eta}$ denotes the tuple $(1/n, \dots, 1/n) \in \mathbb{R}^n$. We write $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in N$. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, $[\cdot]$ and $(\cdot)$ denote permutations such that $x_{[1]} \geq \cdots \geq x_{[n]}$ and $x_{(1)} \leq \cdots \leq x_{(n)}$.

Given $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, some well-known properties of F are the following:

1. Symmetry: $F(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = F(x_1, \dots, x_n)$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and for all permutation σ of N.
2. Monotonicity: $\boldsymbol{x} \geq \boldsymbol{y}$ implies $F(\boldsymbol{x}) \geq F(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.
3. Idempotency: $F(x, \dots, x) = x$ for all $x \in \mathbb{R}$.
4. Compensativeness (or internality): $\min(\boldsymbol{x}) \leq F(\boldsymbol{x}) \leq \max(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
5. Homogeneity of degree 1 (or ratio scale invariance): $F(r\boldsymbol{x}) = rF(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and $r > 0$.

Semi-uninorms, introduced by Liu [24], are a key element in the definition of SUOWA operators. They are monotonic functions with a neutral element in the interval $[0, 1]$ and were introduced as a generalization of uninorms (Yager and Rybalov [25]; see also Mas et al. [26] for an interesting survey on uninorms).

Definition 1. Let $U : [0, 1]^2 \longrightarrow [0, 1]$. U is a semi-uninorm if it is monotonic and possesses a neutral element $e \in [0, 1]$ ($U(e, x) = U(x, e) = x$ for all $x \in [0, 1]$).

Definition 2. Let U be a semi-uninorm.

1. U is right-conjunctive if $U(1, 0) = 0$.
2. U has zero divisors when there are elements $x, y \in (0, 1]$ such that $U(x, y) = 0$.
3. U is a uninorm if it is symmetric and associative ($U(x, U(y, z)) = U(U(x, y), z)$ for all $x, y, z \in [0, 1]$).

Notice that U is right-conjunctive if and only if $U(x, 0) = 0$ for all $x \in [0, 1]$; i.e., 0 is a right absorbing (or annihilator) element of U.

We denote by $\mathcal{U}^e$ (respectively, $\mathcal{U}^e_{\mathrm{i}}$) the set of semi-uninorms (respectively, idempotent semi-uninorms) with neutral element $e \in [0, 1]$. It is worth noting that the semi-uninorms employed in the definition of SUOWA operators have to fulfill two requirements: the neutral element has to be $1/n$ and they have to belong to the following subset (see Llamazares [18]):

$$\widetilde{\mathcal{U}}^{1/n} = \left\{ U \in \mathcal{U}^{1/n} \mid U(1/k, 1/k) \leq 1/k \text{ for all } k \in N \right\}.$$

Obviously $\mathcal{U}_{\mathrm{i}}^{1/n} \subseteq \widetilde{\mathcal{U}}^{1/n}$. Notice that the smallest and the largest elements of $\mathcal{U}_{\mathrm{i}}^{1/n}$ are, respectively, the following uninorms (which were introduced by Yager and Rybalov [25]):

$$U_{\min}(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [1/n, 1]^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

and

$$U_{\max}(x, y) = \begin{cases} \min(x, y) & \text{if } (x, y) \in [0, 1/n]^2, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Apart from the previous ones, several procedures to construct semi-uninorms have been introduced by Llamazares [27] (see also Llamazares [28] for the plot of some semi-uninorms for the case $n = 4$). One of them, which is based on ordinal sums of aggregation operators, allows us to get continuous semi-uninorms. Two of the most relevant continuous semi-uninorms obtained are the following:

$$U_{T_{\mathrm{L}}}(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [1/n, 1]^2, \\ \max(x + y - 1/n, 0) & \text{otherwise,} \end{cases}$$

and

$$U_{\overline{P}}(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [1/n, 1]^2, \\ nxy & \text{otherwise.} \end{cases}$$

In addition to the above, another interesting idempotent semi-uninorm is the following (see Llamazares [29, 30]):

$$U_{\min}^{\max}(x, y) = \begin{cases} \min(x, y) & \text{if } y < 1/n \ \text{ or } \ (x, y) \in [0, 1/n] \times \{1/n\}, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

3. Choquet integral

The Choquet integral has become in the last years a useful tool in several fields, due mainly to its simplicity, versatility and good properties. In its definition (see, for instance, Choquet [31] and Denneberg [32]) it is essential the concept of capacity (Choquet [31]). The notion of capacity is similar to that of probability measure, where the additivity property is changed by monotonicity. And a game is a generalization of a capacity where the monotonicity is ruled out.

Definition 3.

1. A game υ on N is a set function, $\upsilon : 2^N \longrightarrow \mathbb{R}$ satisfying $\upsilon(\varnothing) = 0$.
2. A capacity μ on N is a game on N satisfying $\mu(A) \le \mu(B)$ whenever $A \subseteq B$. In particular, it follows that $\mu : 2^N \longrightarrow [0, \infty)$. A capacity μ is said to be normalized if $\mu(N) = 1$.

In some applications it is very interesting to get a capacity from a game. This can be achieved through the monotonic cover of the game, which is the smallest capacity that contains it (see Maschler and Peleg [33] and Maschler et al. [34]).

Definition 4. Let υ be a game on N. The monotonic cover of υ is the set function $\hat{\upsilon}$ given by

$$\hat{\upsilon}(A) = \max_{B \subseteq A} \upsilon(B).$$

By construction $\hat{\upsilon}$ is a capacity, and when υ is a capacity, $\hat{\upsilon} = \upsilon$. Moreover, if $\upsilon(A) \le 1$ for all $A \subseteq N$ and $\upsilon(N) = 1$, then $\hat{\upsilon}$ is a normalized capacity.

The Choquet integral is usually defined as a functional. However, when we deal with the discrete case and once the capacity has been chosen, it can be seen as an aggregation function (see, for instance, Grabisch et al. [21, p. 181]). This approach is followed in this paper. Moreover, by analogy with the original definition of OWA operators given by Yager [7], we represent it by using a nonincreasing sequences of values (see, for instance, Torra [35] and Llamazares [18]):

Definition 5. Let μ be a capacity on N. The Choquet integral with respect to μ is the function $C_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu(A_{[i]})(x_{[i]} - x_{[i+1]}), \tag{1}$$

where $A_{[i]} = \{[1], \dots, [i]\}$, and we use the convention $x_{[n+1]} = 0$.

The Choquet integral can also be expressed by showing explicitly the weights of the components $x_{[i]}$:

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \left(\mu(A_{[i]}) - \mu(A_{[i-1]})\right) x_{[i]}, \tag{2}$$

where we use the convention $A_{[0]} = \varnothing$. It is worth noting that the Choquet integral possesses properties which are useful in certain information aggregation contexts (see, for instance, Grabisch et al. [21, pp. 192–196]).

Remark 1. Let μ be a capacity on N. Then C_μ is continuous, monotonic and homogeneous of degree 1. Moreover, it is idempotent and compensative when μ is a normalized capacity.

Remark 2. Let μ_1 and μ_2 be two capacities on N. Then $\mu_1 \le \mu_2$ if and only if $C_{\mu_1} \le C_{\mu_2}$.

In the following subsection we collect some of the most important specific cases of the Choquet integral: weighted means, OWA operators and SUOWA operators.

3.1. Weighted means, OWA operators and SUOWA operators

Weighted means and OWA operators (introduced by Yager [7]) are well-known families of functions in the theory of aggregation operators. Both classes of functions are defined through weight distributions that add up to 1.

Definition 6. A vector $\boldsymbol{q} \in [0,1]^n$ is a weighting vector if $\sum_{i=1}^n q_i = 1$.

The set of all weighting vectors of $\mathbb{R}^n$ will be denoted by $\mathcal{W}$.

Definition 7. Let $\boldsymbol{p} \in \mathcal{W}$. The weighted mean associated with $\boldsymbol{p}$ is the function $M_{\boldsymbol{p}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$M_{\boldsymbol{p}}(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i.$$

Two relevant special cases of weighted means are the arithmetic mean ($p_i = 1/n$ for all $i \in N$) and the kth projection ($p_k = 1$).

Definition 8. Let $\boldsymbol{w} \in \mathcal{W}$. The OWA operator associated with $\boldsymbol{w}$ is the function $O_{\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$O_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{[i]}.$$

As in the case of weighted means, the arithmetic mean is a special case of OWA operators (when $w_i = 1/n$ for all $i \in N$). Likewise, the kth order statistic ($\mathrm{OS}_k(\boldsymbol{x}) = x_{(k)}$) is also a special case of OWA operators when $w_{n-k+1} = 1$.

It is well known that weighted means and OWA operators are Choquet integrals with respect to normalized capacities (see, for instance, Fodor et al. [36], Grabisch [1, 37] or Llamazares [18]).

Remark 3.

1. If $\boldsymbol{p} \in \mathcal{W}$, then the weighted mean $M_{\boldsymbol{p}}$ is the Choquet integral with respect to the normalized capacity $\mu_{\boldsymbol{p}}(A) = \sum_{i\in A} p_i$.
2. If $\boldsymbol{w} \in \mathcal{W}$, then the OWA operator $O_{\boldsymbol{w}}$ is the Choquet integral with respect to the normalized capacity $\mu_{|\boldsymbol{w}|}(A) = \sum_{i=1}^{|A|} w_i$.

So, according to Remark 1, weighted means and OWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1. Moreover, OWA operators are also symmetric.

SUOWA operators were introduced by Llamazares [18] in order to deal with situations where both the importance of values and the importance of information sources have to be taken into account. These functions are also a specific case of the Choquet integral where their capacities are the monotonic cover of games constructed by using semi-uninorms with neutral element $1/n$ and the values of the capacities associated with the weighted means and the OWA operators. Specifically, these games are defined as follows.

Definition 9. Let $\boldsymbol{p}, \boldsymbol{w} \in \mathcal{W}$ and let $U \in \widetilde{\mathcal{U}}^{1/n}$.

1. The game associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U is the set function $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}} : 2^N \longrightarrow \mathbb{R}$ defined by

$$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) = |A|\, U\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, \frac{\mu_{|\boldsymbol{w}|}(A)}{|A|}\right)$$

   if $A \neq \varnothing$, and $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(\varnothing) = 0$.

2. $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$, the monotonic cover of the game $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}$, will be called the capacity associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U.

Notice that $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) \leq 1$ for all $A \subseteq N$ and $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(N) = 1$. Therefore, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is always a normalized capacity.

Definition 10. Let $\boldsymbol{p}, \boldsymbol{w} \in \mathcal{W}$ and let $U \in \widetilde{\mathcal{U}}^{1/n}$. The SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U is the function $S^U_{\boldsymbol{p},\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} s_i x_{[i]},$$

where $s_i = \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]}) - \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i-1]})$ for all $i \in N$, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is the capacity associated with $\boldsymbol{p}, \boldsymbol{w}$ and U, and $A_{[i]} = \{[1], \ldots, [i]\}$ (where we use the convention $A_{[0]} = \varnothing$).

According to expression (1), the SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U can also be written as

$$S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]})(x_{[i]} - x_{[i+1]}). \tag{3}$$

By the choice of $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ we have $S^U_{\boldsymbol{p},\boldsymbol{\eta}} = M_{\boldsymbol{p}}$ and $S^U_{\boldsymbol{\eta},\boldsymbol{w}} = O_{\boldsymbol{w}}$ for any $U \in \widetilde{\mathcal{U}}^{1/n}$. Moreover, by Remark 1 and given that $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity, SUOWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1.

3.2. *Conjunctive/Disjunctive character of Choquet integrals*

The notions of k-conjunctive and k-disjunctive functions[1] were introduced by Marichal [23] for determining the conjunctive/disjunctive character of aggregation (it is worth noting that this classification was extended by Komorníková and Mesiar [38]). In essence, k-conjunctive functions are bounded from above by the kth order statistic whereas k-disjunctive functions are bounded from below by the $(n - k + 1)$th order statistic. Although these concepts can be defined for any function, in this paper we focus on Choquet integrals.

Definition 11. Let $k \in N$ and let μ be a normalized capacity on N.

1. C_μ is k-conjunctive if $C_\mu \leq \mathrm{OS}_k$; i.e., $C_\mu(\boldsymbol{x}) \leq x_{(k)}$ for any $\boldsymbol{x} \in \mathbb{R}^n$.
2. C_μ is k-disjunctive if $C_\mu \geq \mathrm{OS}_{n-k+1}$; i.e., $C_\mu(\boldsymbol{x}) \geq x_{(n-k+1)} = x_{[k]}$ for any $\boldsymbol{x} \in \mathbb{R}^n$.

[1]These concepts were originally introduced as *at most k-intolerant* and *at most k-tolerant* functions.

The set of k-conjunctive Choquet integrals will be denoted by $\mathcal{C}_k$, and the set of k-disjunctive Choquet integrals will be denoted by $\mathcal{D}_k$. Notice that if $k \leq k'$, then $\mathcal{C}_k \subseteq \mathcal{C}_{k'}$ and $\mathcal{D}_k \subseteq \mathcal{D}_{k'}$. Moreover, it is noteworthy that Choquet integrals belonging to the sets $\mathcal{C}_k \setminus \mathcal{C}_{k-1}$ and $\mathcal{D}_k \setminus \mathcal{D}_{k-1}$ are also relevant[2].

The following proposition allows to characterize k-conjunctive and k-disjunctive Choquet integrals through the values taken by the capacity on subsets of a given cardinality (Marichal [23]).

Proposition 1. *Let $k \in N$ and let μ be a normalized capacity on N.*

1. $C_\mu \in \mathcal{C}_k$ *if and only if* $\mu(T) = 0$ *for all* $T \subseteq N$ *such that* $|T| \leq n - k$.
2. $C_\mu \in \mathcal{D}_k$ *if and only if* $\mu(T) = 1$ *for all* $T \subseteq N$ *such that* $|T| \geq k$.

From this proposition, the following corollary is immediate (see Definitions 3.3 and 3.4 in Marichal [23]).

Corollary 1. *Let $k \in N \setminus \{1\}$ and let μ be a normalized capacity on N.*

1. $C_\mu \in \mathcal{C}_k \setminus \mathcal{C}_{k-1}$ *if and only if* $\mu(T) = 0$ *for all* $T \subseteq N$ *such that* $|T| \leq n - k$ *and there exists* $T' \subseteq N$*, with* $|T'| = n - k + 1$*, such that* $\mu(T') \neq 0$.
2. $C_\mu \in \mathcal{D}_k \setminus \mathcal{D}_{k-1}$ *if and only if* $\mu(T) = 1$ *for all* $T \subseteq N$ *such that* $|T| \geq k$ *and there exists* $T' \subseteq N$*, with* $|T'| = k - 1$*, such that* $\mu(T') < 1$.

From the second item of Remark 3 we immediately obtain the following result for OWA operators (see also Grabisch et al. [21, p. 30]).

Remark 4. Let $k \in N$ and $\boldsymbol{w} \in \mathcal{W}$.

1. $O_{\boldsymbol{w}} \in \mathcal{C}_k$ if and only if $w_i = 0$ for $i = 1, \dots, n - k$.
2. $O_{\boldsymbol{w}} \in \mathcal{D}_k$ if and only if $w_i = 0$ for $i = k + 1, \dots, n$.

Moreover, if $k \in N \setminus \{1\}$, then

3. $O_{\boldsymbol{w}} \in \mathcal{C}_k \setminus \mathcal{C}_{k-1}$ if and only if $w_i = 0$ for $i = 1, \dots, n - k$ and $w_{n-k+1} > 0$.
4. $O_{\boldsymbol{w}} \in \mathcal{D}_k \setminus \mathcal{D}_{k-1}$ if and only if $w_k > 0$ and $w_i = 0$ for $i = k + 1, \dots, n$.

An important class of conjunctive and disjunctive OWA operators are the window-OWA operators, which were introduced by Yager [39].

Definition 12. A window-OWA operator is an OWA operator defined by means of a weighting vector of the form

$$w_i = \begin{cases} 1/m & \text{if } i \in \{l, \dots, l + m - 1\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $l, m \in N$, with $l + m \leq n + 1$.

[2] These functions were originally introduced by Marichal [23] as *k-intolerant* and *k-tolerant*.

Notice that the previous OWA operator is located between the order statistics $\mathrm{OS}_{n-l-m+2}$ and OS_{n-l+1}; that is, $O_{\boldsymbol{w}} \in \mathcal{C}_{n-l+1} \cap \mathcal{D}_{l+m-1}$. The set of the weighting vectors corresponding to window-OWA operators will be denoted by $\mathcal{W}_{\mathrm{w}}$.

It is worthy of note that k-conjunctive and k-disjunctive Choquet integrals are infrequent in practice. For this reason, Marichal [23] introduced indices for measuring the degree to which a Choquet integral is k-conjunctive or k-disjunctive.

Definition 13. Let $k \in N \setminus \{n\}$ and let μ be a normalized capacity on N. The k-conjunctiveness and k-disjunctiveness indices for C_μ are defined by[3]

$$\mathrm{conj}_k(C_\mu) = 1 - \frac{1}{n-k} \sum_{t=1}^{n-k} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} \mu(T),$$

$$\mathrm{disj}_k(C_\mu) = \frac{1}{n-k} \sum_{t=k}^{n} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} \mu(T) - \frac{1}{n-k} = \frac{1}{n-k} \sum_{t=k}^{n-1} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} \mu(T).$$

Some well-known indices such as the andness and the orness degrees are obtained from k-conjunctiveness and k-disjunctiveness indices when $k = 1$; that is, $\mathrm{andness}(C_\mu) = \mathrm{conj}_1(C_\mu)$ and $\mathrm{orness}(C_\mu) = \mathrm{disj}_1(C_\mu)$. Moreover, it is worth noting that k-disjunctiveness indices preserve the usual order between Choquet integrals whereas k-conjunctiveness indices reverse it.

Remark 5. Let μ_1 and μ_2 be two normalized capacities on N such that $\mu_1 \leq \mu_2$ (which, by Remark 2 is equivalent to $C_{\mu_1} \leq C_{\mu_2}$). Then, for any $k \in N \setminus \{n\}$, we have $\mathrm{conj}_k(C_{\mu_1}) \geq \mathrm{conj}_k(C_{\mu_2})$ and $\mathrm{disj}_k(C_{\mu_1}) \leq \mathrm{disj}_k(C_{\mu_2})$.

Although k-conjunctiveness and k-disjunctiveness indices provide an interesting information about the behavior of Choquet integrals, closed-form expressions of these indices are only known for few operators (see Grabisch et al. [21, p. 378]). Next we gather the value of these indices for weighted means and OWA operators.[4]

$$\mathrm{conj}_k(M_{\boldsymbol{p}}) = \mathrm{disj}_k(M_{\boldsymbol{p}}) = \frac{n+k-1}{2n},$$

$$\mathrm{conj}_k(O_{\boldsymbol{w}}) = 1 - \frac{1}{n-k} \sum_{i=1}^{n-k} (n+1-k-i) w_i,$$

$$\mathrm{disj}_k(O_{\boldsymbol{w}}) = 1 - \frac{1}{n-k} \sum_{i=k+1}^{n} (i-k) w_i.$$

4. Conjunctive/Disjunctive character of SUOWA operators

In this section we analyze the conjunctive/disjunctive character of SUOWA operators from two points of view. On the one hand, we study the k-conjunctive or k-disjunctive character of SUOWA operators in relation to the corre-

[3] Notice that in $\mathrm{conj}_k(C_\mu)$ the summation begins in $t = 1$ because when $t = 0$ we have $\mu(T) = \mu(\emptyset) = 0$.

[4] Note that in this paper we consider the original definition of OWA operators given by Yager [7], where the components of $\boldsymbol{x}$ are ordered in a nonincreasing way. For this reason, the values of these indices for OWA operators do not match with those shown by Grabisch et al. [21], where the components are ordered in a nondecreasing way.

sponding property of the OWA operators associated with them. On the other hand, we provide bounds or values for k-conjunctiveness and k-disjunctiveness indices of SUOWA operators regarding the corresponding indices of the OWA operators associated with them. In both cases we also show interesting properties related to the convex combination of semi-uninorms.

4.1. *k-conjunctive and k-disjunctive SUOWA operators*

We begin by introducing an attractive property: under certain hypothesis, the k-conjunctive or k-disjunctive character of SUOWA operators is preserved by convex combinations of semi-uninorms.

Proposition 2. *Let $k \in N$, let $\boldsymbol{p}, \boldsymbol{w} \in \mathcal{W}$, let $U_1, \dots, U_m \in \widetilde{\mathcal{U}}^{1/n}$ such that the games $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_1}, \dots, \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_m}$ are normalized capacities, let $\boldsymbol{\lambda}$ be a weighting vector of $\mathbb{R}^m$, and let $U = \sum_{i=1}^{m} \lambda_i U_i$. Then*

1. *If $S_{\boldsymbol{p},\boldsymbol{w}}^{U_1}, \dots, S_{\boldsymbol{p},\boldsymbol{w}}^{U_m} \in \mathcal{C}_k$, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U} \in \mathcal{C}_k$.*
2. *If $S_{\boldsymbol{p},\boldsymbol{w}}^{U_1}, \dots, S_{\boldsymbol{p},\boldsymbol{w}}^{U_m} \in \mathcal{D}_k$, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U} \in \mathcal{D}_k$.*

Moreover, if $k \in N \setminus \{1\}$, then

3. *If $S_{\boldsymbol{p},\boldsymbol{w}}^{U_1}, \dots, S_{\boldsymbol{p},\boldsymbol{w}}^{U_m} \in \mathcal{C}_k \setminus \mathcal{C}_{k-1}$, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U} \in \mathcal{C}_k \setminus \mathcal{C}_{k-1}$.*
4. *If $S_{\boldsymbol{p},\boldsymbol{w}}^{U_1}, \dots, S_{\boldsymbol{p},\boldsymbol{w}}^{U_m} \in \mathcal{D}_k \setminus \mathcal{D}_{k-1}$, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U} \in \mathcal{D}_k \setminus \mathcal{D}_{k-1}$.*

Next we show that it is possible to retain the k-conjunctive character of OWA operators when the semi-uninorm used to construct the SUOWA operator is right-conjunctive.

Proposition 3. *Let $\boldsymbol{w} \in \mathcal{W}$ and $U \in \widetilde{\mathcal{U}}^{1/n}$.*

1. *If $k \in N$, $O_{\boldsymbol{w}} \in \mathcal{C}_k$ and U is right-conjunctive, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U} \in \mathcal{C}_k$ for any weighting vector $\boldsymbol{p}$.*
2. *If $k \in N \setminus \{1\}$, $O_{\boldsymbol{w}} \in \mathcal{C}_k \setminus \mathcal{C}_{k-1}$ and U is right-conjunctive with no zero divisors, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U} \in \mathcal{C}_k \setminus \mathcal{C}_{k-1}$ for any weighting vector $\boldsymbol{p}$ such that $|\{i \in N \mid p_i > 0\}| \geq k$.*

Given that $U_{\min}$, $U_{\widetilde{P}}$, and $U_{\min}^{\max}$ are right-conjunctive with no zero divisors, the statements of Proposition 3 are satisfied when we consider these semi-uninorms.

Analogously, it is also possible to maintain the k-disjunctive character of the OWA operator if the semi-uninorm associated with the SUOWA operator satisfies certain conditions (for instance, $U_{\max}$ and $U_{\min}^{\max}$ meet these conditions).

Proposition 4. *Let $\boldsymbol{w} \in \mathcal{W}$ and $U \in \widetilde{\mathcal{U}}^{1/n}$.*

1. *If $k \in N$, $O_{\boldsymbol{w}} \in \mathcal{D}_k$ and $U(x, y) \geq \max(x, y)$ for all $y > 1/n$, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U} \in \mathcal{D}_k$ for any weighting vector $\boldsymbol{p}$.*
2. *If $k \in N \setminus \{1\}$, $O_{\boldsymbol{w}} \in \mathcal{D}_k \setminus \mathcal{D}_{k-1}$ and $U(x, y) = \max(x, y)$ for all $y > 1/n$, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U} \in \mathcal{D}_k \setminus \mathcal{D}_{k-1}$ for any weighting vector $\boldsymbol{p}$ such that $|\{i \in N \mid p_i > 0\}| \geq k$.*

It is worth noting that the SUOWA operators constructed by using the semi-uninorm $U_{\min}^{\max}$ preserve at the same time the conjunctive/disjunctive character of the OWA operator associated with them. In this way, it is possible to get operators located between two order statistics that take into account the weights of the information sources. Obviously, when the OWA operator is an order statistic, the SUOWA operator coincides with it. Moreover, in the particular case of considering weighting vectors associated with window-OWA operators, the games associated with this semi-uninorm are normalized capacities.

Proposition 5. *Let $\boldsymbol{w} \in \mathcal{W}_{\mathrm{w}}$. Then, for any weighting vector $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$ is a normalized capacity on N.*

4.2. k-conjunctiveness and k-disjunctiveness indices of SUOWA operators

The first result of this subsection shows that, when we consider convex combination of semi-uninorms and the games associated with these semi-uninorms are normalized capacities, then the k-conjunctiveness and k-disjunctiveness indices of the SUOWA operator associated with the constructed semi-uninorm can be obtained through the same convex combination of the indices of the SUOWA operators associated with the former semi-uninorms.

Proposition 6. *Let $\boldsymbol{p}, \boldsymbol{w} \in \mathcal{W}$, let $U_1, \dots, U_m \in \widetilde{\mathcal{U}}^{1/n}$ such that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_1}, \dots, \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_m}$ be normalized capacities, let $\boldsymbol{\lambda}$ be a weighting vector, and let $U = \sum_{i=1}^{m} \lambda_i U_i$. Then, for any $k \in N \setminus \{n\}$, we have*

$$\operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U}\right) = \sum_{i=1}^{m} \lambda_i \operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_i}\right), \qquad \operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U}\right) = \sum_{i=1}^{m} \lambda_i \operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_i}\right).$$

In the case of the semi-uninorm $U_{\min}$, the following result shows that if the weighting vector $\boldsymbol{w}$ meets a certain condition, then the k-conjunctiveness (k-disjunctiveness) indices of the OWA operators are lower (upper) bounds for the corresponding indices of the SUOWA operators.

Proposition 7. *Let $\boldsymbol{w} \in \mathcal{W}$ such that $\sum_{i=1}^{j} w_i < j/n$ for all $j \in N \setminus \{n\}$. Then, for any weighting vector $\boldsymbol{p}$ and any $k \in N \setminus \{n\}$, we have*

$$\operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}\right) \geq \operatorname{conj}_k(O_{\boldsymbol{w}}), \qquad \operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}\right) \leq \operatorname{disj}_k(O_{\boldsymbol{w}}).$$

A similar result can be found for the uninorm $U_{\max}$ and weighting vectors $\boldsymbol{w}$ such that $\sum_{i=1}^{j} w_i > j/n$ for all $j \in N$.

Proposition 8. *Let $\boldsymbol{w} \in \mathcal{W}$ such that $\sum_{i=1}^{j} w_i > j/n$ for all $j \in N \setminus \{n\}$. Then, for any weighting vector $\boldsymbol{p}$ and any $k \in N \setminus \{n\}$, we have*

$$\operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}\right) \leq \operatorname{conj}_k(O_{\boldsymbol{w}}), \qquad \operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}\right) \geq \operatorname{disj}_k(O_{\boldsymbol{w}}).$$

In the case of the semi-uninorms $U_{T_{\mathrm{L}}}$ and $U_{\widetilde{P}}$, and under certain hypothesis on the weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$, the values of the k-conjunctiveness and k-disjunctiveness indices of SUOWA operators coincide with the respective indices of OWA operators.

Proposition 9. *Let $\boldsymbol{p}, \boldsymbol{w} \in \mathcal{W}$ such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$ and $\min_{i \in N} p_i + \min_{i \in N} w_i \geq 1/n$. Then, for any $k \in N \setminus \{n\}$, we have*

$$\operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right) = \operatorname{conj}_k(O_{\boldsymbol{w}}), \qquad \operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right) = \operatorname{disj}_k(O_{\boldsymbol{w}}).$$

Proposition 10. *Let $\boldsymbol{w} \in \mathcal{W}$ such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$. If $\boldsymbol{p} \in \mathcal{W}$ is such that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a capacity on N, then, for any $k \in N \setminus \{n\}$, we have*

$$\operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}\right) = \operatorname{conj}_k(O_{\boldsymbol{w}}), \qquad \operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}\right) = \operatorname{disj}_k(O_{\boldsymbol{w}}).$$

It is worth noting that the conditions $\sum_{i=1}^{j} w_i \leq j/n$ required in Propositions 9 and 10 can be guaranteed when the weighting vector $\boldsymbol{w}$ is a nondecreasing sequence of weights; i.e., $w_1 \leq w_2 \leq \cdots \leq w_n$. Moreover, in this case, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a normalized capacity on N for any weighting vector $\boldsymbol{p}$ (see Llamazares [22]). Notice that nondecreasing sequence of weights appear frequently in the literature; for instance, when the weights form a (nondecreasing) arithmetic progression, which arise in the 2-additive symmetric normalized capacities (see Beliakov et al. [40, p. 86] and Bortot and Marques Pereira [41]) and in some models proposed in the literature to determine the OWA weighting vector[5] (see Liu [45]). Likewise, nondecreasing weights allow to characterize the Schur-concavity of OWA operators (see Bortot and Marques Pereira [46][6]).

5. Example

Suppose that the Department of Mathematics in a Faculty of Economics offers a research assistantship for the students accepted into the M. Sc. in Economics. Applicants are evaluated with respect to seven subjects: Mathematics I (MatI), Mathematics II (MatII), Mathematics III (MatIII), Statistics I (StaI), Statistics II (StaII), Econometrics I (EcoI), and Econometrics II (EcoII), and the members of the committee would like to take into account the following aspects:

1. Each one of the first three subjects is considered twice as important as each one of the remaining four.
2. Since the same subject has been given in several groups by different professors, extreme marks should be discarded to avoid bias.

Notice that the first point corresponds to a weighted mean type aggregation by using the weighting vector $\boldsymbol{p} = (0.2, 0.2, 0.2, 0.1, 0.1, 0.1, 0.1)$ whereas the second point corresponds to an OWA type aggregation. We consider the following weighting vectors: $\boldsymbol{w} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$ (i.e., the maximum and minimum marks are ruled out),

[5]Note that this topic has generated an abundant literature in the past years; see, for instance, Troiano and Yager [42], Liu [43], and Bai et al. [44].

[6]Notice that in Bortot and Marques Pereira [46] appears Schur-convexity instead of Schur-concavity because in the definition of OWA operators these authors consider that the components of $\boldsymbol{x}$ are ordered in a nondecreasing way.

$\boldsymbol{w} = (0, 0, 1/3, 1/3, 1/3, 0, 0)$ (i.e., the two highest and lowest marks are discarded), and $\boldsymbol{w} = (0, 0, 0, 1, 0, 0, 0)$ (i.e., the median is used).

In Table 1 we collect the marks obtained by three students (marks are given on a scale from 0 to 10). Notice that in the subject Statistics I there is a big difference between the marks obtained by the students: student A gets its highest mark whereas students B and C get their lowest grades. Moreover, given the difference with the marks obtained by the three students in the remaining subjects, these values could be considered as outliers.

Table 1: Marks of the students.

Student	MatI	MatII	MatIII	StaI	StaII	EcoI	EcoII
A	7.9	7.8	7.7	9.8	7.5	7.6	7.4
B	7.7	7.8	7.9	5.2	8.3	8.4	8.5
C	8.2	8.4	8.5	5.2	7.7	7.8	7.9

Table 2 gathers the global evaluation given to the three students by the weighted mean, the OWA operators and three idempotent SUOWA operators: $S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$, $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$, and $S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$. As you can appreciate, the chosen students with the weighted mean and the OWA operators are, respectively, A and B (although there is a tie between B and C when the median is used). In the case of the SUOWA operators considered here we have the following results:

1. When $\boldsymbol{w} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$, $S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$, $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$, and $S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$ select student C. In fact, this student is chosen for any SUOWA operator associated with an idempotent semi-uninorm U: Given that $S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}} \leq S^{U}_{\boldsymbol{p},\boldsymbol{w}} \leq S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$ for any idempotent semi-uninorm U (see Llamazares [18]), we get
$$S^{U}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}_{\mathrm{A}}) \leq S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}_{\mathrm{A}}) = 7.73 < 8.01 = S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}_{\mathrm{C}}) \leq S^{U}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}_{\mathrm{C}}),$$
$$S^{U}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}_{\mathrm{B}}) \leq S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}_{\mathrm{B}}) = 7.98 < 8.01 = S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}_{\mathrm{C}}) \leq S^{U}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}_{\mathrm{C}}).$$
2. When $\boldsymbol{w} = (0, 0, 1/3, 1/3, 1/3, 0, 0)$, $S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$ and $S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$ choose student C while $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$ picks out student B.
3. When $\boldsymbol{w} = (0, 0, 0, 1, 0, 0, 0)$, the three SUOWA operators select student C, although there is a tie between B and C when the semi-uninorm $U^{\max}_{\min}$ is considered (notice that $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$ coincides with $O_{\boldsymbol{w}}$ when $\boldsymbol{w} = (0, 0, 0, 1, 0, 0, 0)$).

It is worth noting that the selection of student C seems the most appropriate if we want to take into account both the importance of the subjects and the lack of bias. Notice that the SUOWA operators considered here choose this student on most occasions.

In Tables 3–5 we show the weights of the subjects for the students and the SUOWA operators considered (subjects are ordered according to the marks obtained in each of them by the students, from highest to lowest). By the results of the previous section we know that:

1. $S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$, $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}} \in \mathcal{C}_6$ and $S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$, $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}} \in \mathcal{D}_6$, given that $O_{\boldsymbol{w}} \in \mathcal{C}_6 \cap \mathcal{D}_6$ when $\boldsymbol{w} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$.

Table 2: Global evaluation of the students by using some aggregation operators.

Student	M_p	$w = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$				$w = (0, 0, 1/3, 1/3, 1/3, 0, 0)$				$w = (0, 0, 0, 1, 0, 0, 0)$			
		O_w	$S_{p,w}^{U_{\min}}$	$S_{p,w}^{U_{\min}^{\max}}$	$S_{p,w}^{U_{\max}}$	O_w	$S_{p,w}^{U_{\min}}$	$S_{p,w}^{U_{\min}^{\max}}$	$S_{p,w}^{U_{\max}}$	O_w	$S_{p,w}^{U_{\min}}$	$S_{p,w}^{U_{\min}^{\max}}$	$S_{p,w}^{U_{\max}}$
A	7.91	7.7	7.71	7.71	7.73	7.7	$7.70\overline{3}$	$7.70\overline{3}$	7.75	7.7	7.7	7.7	7.78
B	7.72	8.02	7.96	7.98	7.98	8	7.94	$7.98\overline{6}$	$7.98\overline{6}$	7.9	7.85	7.9	7.9
C	7.88	8	8.01	8.01	8.13	$7.9\overline{6}$	7.97	7.97	8.15	7.9	7.9	7.9	8.18

2. $S_{p,w}^{U_{\min}}, S_{p,w}^{U_{\min}^{\max}} \in \mathcal{C}_5$ and $S_{p,w}^{U_{\max}}, S_{p,w}^{U_{\min}^{\max}} \in \mathcal{D}_5$, given that $O_w \in \mathcal{C}_5 \cap \mathcal{D}_5$ when $w = (0, 0, 1/3, 1/3, 1/3, 0, 0)$.
3. $S_{p,w}^{U_{\min}}, S_{p,w}^{U_{\min}^{\max}} \in \mathcal{C}_4$ and $S_{p,w}^{U_{\max}}, S_{p,w}^{U_{\min}^{\max}} \in \mathcal{D}_4$, given that $O_w \in \mathcal{C}_4 \cap \mathcal{D}_4$ when $w = (0, 0, 0, 1, 0, 0, 0)$.

Therefore,

1. When $w = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$, the weights of the subjects with the highest marks are null for $S_{p,w}^{U_{\min}}$, the weights of the subjects with the lowest marks are null for $S_{p,w}^{U_{\max}}$, and both conditions are fulfilled for $S_{p,w}^{U_{\min}^{\max}}$.
2. When $w = (0, 0, 1/3, 1/3, 1/3, 0, 0)$, the weights of the subjects with the two highest marks are null for $S_{p,w}^{U_{\min}}$, the weights of the subjects with the two lowest marks are null for $S_{p,w}^{U_{\max}}$, and both conditions are fulfilled for $S_{p,w}^{U_{\min}^{\max}}$.
3. When $w = (0, 0, 0, 1, 0, 0, 0)$, the weights of the subjects with the three highest marks are null for $S_{p,w}^{U_{\min}}$, the weights of the subjects with the three lowest marks are null for $S_{p,w}^{U_{\max}}$, and both conditions are fulfilled for $S_{p,w}^{U_{\min}^{\max}}$.

Tables[7] 6 and 7 collect the k-conjunctiveness and k-disjunctiveness indices of the weighted mean, the OWA operator and the three SUOWA operators considered. Note that, when $w = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$, the operators O_w, $S_{p,w}^{U_{\min}}$, and $S_{p,w}^{U_{\min}^{\max}}$ belong to $\mathcal{C}_6$ and, therefore,

$$\mathrm{conj}_6(O_w) = \mathrm{conj}_6\left(S_{p,w}^{U_{\min}}\right) = \mathrm{conj}_6\left(S_{p,w}^{U_{\min}^{\max}}\right) = 1.$$

Analogously, since $O_w, S_{p,w}^{U_{\min}^{\max}}, S_{p,w}^{U_{\max}} \in \mathcal{D}_6$, we get

$$\mathrm{disj}_6(O_w) = \mathrm{disj}_6\left(S_{p,w}^{U_{\min}^{\max}}\right) = \mathrm{disj}_6\left(S_{p,w}^{U_{\max}}\right) = 1.$$

[7]The data of these tables have been rounded to three decimal places.

Table 3: Weights of the subjects for the student A.

	$\boldsymbol{w} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$			$\boldsymbol{w} = (0, 0, 1/3, 1/3, 1/3, 0, 0)$			$\boldsymbol{w} = (0, 0, 0, 1, 0, 0, 0)$		
	$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$
StaI	0	0	0	0	0	0	0	0	0
MatI	0.2	0.2	0.3	0	0	0.3	0	0	0.3
MatII	0.2	0.2	0.2	$0.\overline{3}$	$0.\overline{3}$	0.2	0	0	0.2
MatIII	0.3	0.3	0.2	$0.3\overline{6}$	$0.3\overline{6}$	0.2	1	1	0.5
EcoI	0.1	0.1	0.1	0.3	0.3	0.3	0	0	0
StaII	0.2	0.2	0.2	0	0	0	0	0	0
EcoII	0	0	0	0	0	0	0	0	0

Table 4: Weights of the subjects for the student B.

	$\boldsymbol{w} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$			$\boldsymbol{w} = (0, 0, 1/3, 1/3, 1/3, 0, 0)$			$\boldsymbol{w} = (0, 0, 0, 1, 0, 0, 0)$		
	$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$
EcoII	0	0	0	0	0	0	0	0	0
EcoI	0.2	0.2	0.2	0	0	0	0	0	0
StaII	0.1	0.1	0.1	0.3	0.3	0.3	0	0	0
MatIII	0.2	0.3	0.3	0.2	$0.3\overline{6}$	$0.3\overline{6}$	0.5	1	1
MatII	0.2	0.2	0.2	0.2	$0.\overline{3}$	$0.\overline{3}$	0.5	0	0
MatI	0.3	0.2	0.2	0.3	0	0	0	0	0
StaI	0	0	0	0	0	0	0	0	0

For similar reasons as above, when $\boldsymbol{w} = (0, 0, 1/3, 1/3, 1/3, 0, 0)$ we get

$$\begin{aligned}
\mathrm{conj}_5(O_{\boldsymbol{w}}) &= \mathrm{conj}_5\left(S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \mathrm{conj}_5\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1,\\
\mathrm{conj}_6(O_{\boldsymbol{w}}) &= \mathrm{conj}_6\left(S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \mathrm{conj}_6\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1,\\
\mathrm{disj}_5(O_{\boldsymbol{w}}) &= \mathrm{disj}_5\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \mathrm{disj}_5\left(S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1,\\
\mathrm{disj}_6(O_{\boldsymbol{w}}) &= \mathrm{disj}_6\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \mathrm{disj}_6\left(S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1,
\end{aligned}$$

Table 5: Weights of the subjects for the student C.

	$\boldsymbol{w} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$			$\boldsymbol{w} = (0, 0, 1/3, 1/3, 1/3, 0, 0)$			$\boldsymbol{w} = (0, 0, 0, 1, 0, 0, 0)$		
	$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$
MatIII	0	0	0.2	0	0	0.2	0	0	0.2
MatII	0.2	0.2	0.2	0	0	0.2	0	0	0.2
MatI	0.2	0.2	0.2	$0.\overline{3}$	$0.\overline{3}$	0.2	0	0	0.2
EcoII	0.3	0.3	0.1	$0.3\overline{6}$	$0.3\overline{6}$	0.1	1	1	0.4
EcoI	0.1	0.1	0.1	0.3	0.3	0.3	0	0	0
StaII	0.2	0.2	0.2	0	0	0	0	0	0
StaI	0	0	0	0	0	0	0	0	0

and when $\boldsymbol{w} = (0, 0, 0, 1, 0, 0, 0)$ we have

$$\begin{aligned}
\operatorname{conj}_4(O_{\boldsymbol{w}}) &= \operatorname{conj}_4\left(S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \operatorname{conj}_4\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1,\\
\operatorname{conj}_5(O_{\boldsymbol{w}}) &= \operatorname{conj}_5\left(S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \operatorname{conj}_5\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1,\\
\operatorname{conj}_6(O_{\boldsymbol{w}}) &= \operatorname{conj}_6\left(S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \operatorname{conj}_6\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1,\\
\operatorname{disj}_4(O_{\boldsymbol{w}}) &= \operatorname{disj}_4\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \operatorname{disj}_4\left(S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1,\\
\operatorname{disj}_5(O_{\boldsymbol{w}}) &= \operatorname{disj}_5\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \operatorname{disj}_5\left(S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1,\\
\operatorname{disj}_6(O_{\boldsymbol{w}}) &= \operatorname{disj}_6\left(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \operatorname{disj}_6\left(S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}\right) = 1.
\end{aligned}$$

Likewise, it is worth mentioning that in this example the k-conjunctiveness and k-disjunctiveness indices of $O_{\boldsymbol{w}}$ and $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$ are identical except in the case of $k = 4$ and $\boldsymbol{w} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$ or $\boldsymbol{w} = (0, 0, 1/3, 1/3, 1/3, 0, 0)$. So, it seems that $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$ shares many characteristics with the OWA operator with the advantage of allowing to take into account the weights of the subjects.

6. Concluding remarks

Weighted means and OWA operators, which are well-known families of functions widely used in several fields, can be generalized by means of SUOWA operators. All of them are specific cases of Choquet integrals, so they possess interesting properties such as continuity, monotonicity, idempotency, compensativeness, and homogeneity of degree 1. In addition to knowing that they satisfy these properties, it is also relevant to know the behavior of these operators in relation to some indices proposed in the literature.

Table 6: Conjunctiveness indices for some aggregation operators.

	M_p	$\boldsymbol{w} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$				$\boldsymbol{w} = (0, 0, 1/3, 1/3, 1/3, 0, 0)$				$\boldsymbol{w} = (0, 0, 0, 1, 0, 0, 0)$			
		$O_{\boldsymbol{w}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$	$O_{\boldsymbol{w}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$	$O_{\boldsymbol{w}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$
conj_1	0.5	0.5	0.535	0.5	0.465	0.5	0.563	0.5	0.437	0.5	0.541	0.5	0.39
conj_2	0.571	0.6	0.625	0.6	0.558	0.6	0.659	0.6	0.524	0.6	0.649	0.6	0.469
conj_3	0.643	0.7	0.71	0.7	0.647	0.75	0.766	0.75	0.655	0.75	0.797	0.75	0.586
conj_4	0.714	0.8	0.804	0.804	0.733	0.889	0.89	0.89	0.764	1	1	1	0.781
conj_5	0.786	0.9	0.9	0.9	0.814	1	1	1	0.843	1	1	1	0.843
conj_6	0.857	1	1	1	0.914	1	1	1	0.914	1	1	1	0.914

Table 7: Disjunctiveness indices for some aggregation operators.

	M_p	$\boldsymbol{w} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$				$\boldsymbol{w} = (0, 0, 1/3, 1/3, 1/3, 0, 0)$				$\boldsymbol{w} = (0, 0, 0, 1, 0, 0, 0)$			
		$O_{\boldsymbol{w}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$	$O_{\boldsymbol{w}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$	$O_{\boldsymbol{w}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$
disj_1	0.5	0.5	0.465	0.5	0.535	0.5	0.437	0.5	0.563	0.5	0.459	0.5	0.61
disj_2	0.571	0.6	0.558	0.6	0.625	0.6	0.524	0.6	0.659	0.6	0.551	0.6	0.714
disj_3	0.643	0.7	0.647	0.7	0.71	0.75	0.655	0.75	0.766	0.75	0.689	0.75	0.836
disj_4	0.714	0.8	0.733	0.804	0.804	0.889	0.764	0.89	0.89	1	0.918	1	1
disj_5	0.786	0.9	0.814	0.9	0.9	1	0.843	1	1	1	0.971	1	1
disj_6	0.857	1	0.914	1	1	1	0.914	1	1	1	1	1	1

In this paper we have focused on the conjunctive/disjunctive character of SUOWA operators and we have established conditions under which the k-conjunctive or k-disjunctive character of an OWA operator is retained by the SUOWA operators associated with it. Likewise, we have showed that, for some semi-uninorms and under certain hypothesis, the k-conjunctiveness and k-disjunctiveness indices of SUOWA operators coincide with the corresponding indices of the OWA operators associated with them. Of particular interest are the results obtained when we consider the semi-uninorm $U_{\min}^{\max}$, given that we can get operators located between two order statistics that take into account the weights of the information sources. As far as we know, they are the only functions in the literature fulfilling this characteristic.

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Appendix A. Proofs

In the proofs of some results we will use the following proposition (see Llamazares [27])

Proposition 11. *Let* $\boldsymbol{p}, \boldsymbol{w} \in \mathcal{W}$*, let* $U_1, \ldots, U_m \in \widetilde{\mathcal{U}}^{1/n}$ *such that* $\upsilon^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \ldots, \upsilon^{U_m}_{\boldsymbol{p},\boldsymbol{w}}$ *be normalized capacities, let* $\boldsymbol{\lambda}$ *be a weighting vector of* $\mathbb{R}^m$*, and let* $U = \sum_{i=1}^m \lambda_i U_i$*. Then* $\upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}$ *is a normalized capacity defined by*

$$\upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}(T) = \sum_{i=1}^{m} \lambda_i \upsilon^{U_i}_{\boldsymbol{p},\boldsymbol{w}}(T),$$

for any subset T *of* N*.*

Proof of Proposition 2. It is immediate from Propositions 11 and 1, and Corollary 1. □

Proof of Proposition 3.

1. By the first item of Proposition 1, it is sufficient to prove that $\hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(T) = 0$ for any $T \subseteq N$ such that $1 \le |T| \le n-k$. Since $O_{\boldsymbol{w}}$ is k-conjunctive, we have $\mu_{|\boldsymbol{w}|}(A) = 0$ for any $A \subseteq N$ such that $1 \le |A| \le n-k$. Therefore, because U is right-conjunctive, we get

   $$\upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}(A) = |A|\, U\!\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, 0\right) = 0$$

   for any $A \subseteq N$ such that $1 \le |A| \le n-k$. Consequently, given $T \subseteq N$ such that $|T| \le n-k$, we have

   $$\hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(T) = \max_{A \subseteq T} \upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}(A) = 0.$$

2. By the first item of Corollary 1 and the first item of this proposition, it is sufficient to prove that there exists $T \subseteq N$, with $|T| = n-k+1$, such that $\hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(T) > 0$. Let $\boldsymbol{p}$ be a weighting vector such that $|\{i \in N \mid p_i > 0\}| \ge k$ (and, consequently, $|\{i \in N \mid p_i = 0\}| \le n-k$); and let $T \subseteq N$ such that $|T| = n-k+1$. Since $O_{\boldsymbol{w}}$ is k-conjunctive but not $(k-1)$-conjunctive, by the third item of Remark 4 we have $w_{n-k+1} > 0$. Therefore, as U is without zero divisors, we get

   $$\upsilon^{U}_{\boldsymbol{p},\boldsymbol{w}}(T) = |T|\, U\!\left(\frac{\mu_{\boldsymbol{p}}(T)}{|T|}, \frac{\mu_{|\boldsymbol{w}|}(T)}{|T|}\right) > 0,$$

   and, consequently, $\hat{\upsilon}^{U}_{\boldsymbol{p},\boldsymbol{w}}(T) > 0$. □

Proof of Proposition 4.

1. By the second item of Proposition 1, it is sufficient to prove that $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(T) = 1$ for any $T \subseteq N$ such that $|T| \geq k$. Given that $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(N) = 1$, it is sufficient to consider $T \subseteq N$ such that $k \leq |T| \leq n-1$. Since $O_{\boldsymbol{w}}$ is k-disjunctive, we have $\mu_{|\boldsymbol{w}|}(A) = 1$ for any $A \subseteq N$ such that $|A| \geq k$. Therefore, as $U(x,y) \geq \max(x,y)$ for all $y > 1/n$, we get

$$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(T) = |T|\, U\left(\frac{\mu_{\boldsymbol{p}}(T)}{|T|}, \frac{1}{|T|}\right) \geq \max\left(\mu_{\boldsymbol{p}}(T), 1\right) = 1,$$

and, consequently, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(T) \geq \upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(T) = 1$.

2. By the second item of Corollary 1 and the first item of this proposition, it is sufficient to prove that there exists $T \subseteq N$, with $|T| = k-1$, such that $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(T) < 1$. Let $\boldsymbol{p}$ be a weighting vector such that $|\{i \in N \mid p_i > 0\}| \geq k$. Since $O_{\boldsymbol{w}}$ is k-disjunctive but not $(k-1)$-disjunctive, by the fourth item of Remark 4 we have $w_k > 0$. Therefore, as $U(x,y) \leq \max(x,y)$, we get

$$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) = |A|\, U\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, \frac{\mu_{|\boldsymbol{w}|}(A)}{|A|}\right) \leq \max\left(\sum_{i\in A} p_i, \sum_{i=1}^{k-1} w_i\right) < 1$$

for any $A \subseteq N$ such that $|A| \leq k-1$. Consequently, given $T \subseteq N$ such that $|T| = k-1$, we have

$$\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(T) = \max_{A\subseteq T} \upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) < 1. \qquad \square$$

In the proof of Proposition 5 we will use the following lemma.

Lemma 1. *Let $\boldsymbol{w} \in \mathcal{W}_{\mathrm{w}}$, $\boldsymbol{w} \neq \boldsymbol{\eta}$. If $\sum_{i=1}^{p} w_i \geq p/n$, then $\sum_{i=1}^{q} w_i > q/n$ for any $q \in \{p+1, \dots, n-1\}$.*

Proof. Given that $\boldsymbol{w} \in \mathcal{W}_{\mathrm{w}}$ and $\boldsymbol{w} \neq \boldsymbol{\eta}$, there exist $l, m \in N$, with $m < n$ and $l + m \leq n+1$ such that

$$w_i = \begin{cases} 1/m & \text{if } i \in \{l, \dots, l+m-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sum_{i=1}^{p} w_i \geq p/n$, we have $p \geq l$. We distinguish two cases:

1. If $q \in \{p+1, \dots, l+m-1\}$, then

$$\sum_{i=1}^{q} w_i = \sum_{i=1}^{p} w_i + (q-p)\frac{1}{m} > \frac{p}{n} + (q-p)\frac{1}{n} = \frac{q}{n}.$$

2. If $q \in \{l+m, \dots, n-1\}$, then $\sum_{i=1}^{q} w_i = 1 > q/n$. $\square$

Proof of Proposition 5. Let $\boldsymbol{w} \in \mathcal{W}_{\mathrm{w}}$. Since the case $\boldsymbol{w} = \boldsymbol{\eta}$ is obvious ($\upsilon^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{\eta}} = \mu_{\boldsymbol{p}}$ for any weighting vector $\boldsymbol{p}$), we suppose $\boldsymbol{w} \neq \boldsymbol{\eta}$. Let $\boldsymbol{p} \in \mathcal{W}$, and consider $A \subseteq B \subseteq N$. We are going to show that $\upsilon^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A) \leq \upsilon^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(B)$. Notice that the cases $A = B$ and $B = N$ are trivial. So, consider $A \subsetneq B \subsetneq N$. We distinguish two cases:

1. If $\mu_{|\boldsymbol{w}|}(A)/\,|A| < 1/n$, then

$$\upsilon^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(A) = \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) \leq \min\left(\sum_{i\in B} p_i, \sum_{i=1}^{|B|} w_i\right) \leq \upsilon^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(B).$$

2. If $\mu_{|\boldsymbol{w}|}(A)/\,|A| \geq 1/n$, then, by Lemma 1, we have

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}(A) \leq \max\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) \leq \max\left(\sum_{i\in B} p_i, \sum_{i=1}^{|B|} w_i\right) = \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}(B). \qquad \square$$

Proof of Proposition 6. Let $k \in \{1, \ldots, n-1\}$. According to Proposition 11, we have

$$\begin{aligned}
\operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U}\right) &= \sum_{i=1}^{m} \lambda_i - \frac{1}{n-k} \sum_{t=1}^{n-k} \frac{1}{\binom{n}{t}} \sum_{\substack{T\subseteq N\\|T|=t}} \left(\sum_{i=1}^{m} \lambda_i \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_i}(T)\right)\\
&= \sum_{i=1}^{m} \lambda_i \left(1 - \frac{1}{n-k} \sum_{t=1}^{n-k} \frac{1}{\binom{n}{t}} \sum_{\substack{T\subseteq N\\|T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_i}(T)\right) = \sum_{i=1}^{m} \lambda_i \operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_i}\right).
\end{aligned}$$

Analogously,

$$\begin{aligned}
\operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U}\right) &= \frac{1}{n-k} \sum_{t=k}^{n-1} \frac{1}{\binom{n}{t}} \sum_{\substack{T\subseteq N\\|T|=t}} \left(\sum_{i=1}^{m} \lambda_i \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_i}(T)\right)\\
&= \sum_{i=1}^{m} \lambda_i \left(\frac{1}{n-k} \sum_{t=k}^{n-1} \frac{1}{\binom{n}{t}} \sum_{\substack{T\subseteq N\\|T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_i}(T)\right) = \sum_{i=1}^{m} \lambda_i \operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_i}\right). \qquad \square
\end{aligned}$$

Proof of Proposition 7. Under the hypothesis of Proposition 7 we have $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}} \leq O_{\boldsymbol{w}}$ (see Llamazares [27]). Therefore, taking into account Remark 5, we get the result. □

Proof of Proposition 8. Under the hypothesis of Proposition 8 we have $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}} \geq O_{\boldsymbol{w}}$ (see Llamazares [27]). Therefore, taking into account Remark 5, we get the result. □

The following lemma will be used in the proof of Proposition 9 (see Llamazares [22] for a proof of this lemma).

Lemma 2. *Let* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *be two weighting vectors such that* $\sum_{i=1}^{j} w_i \leq j/n$ *for all* $j \in N$ *and* $\min_{i\in N} p_i + \min_{i\in N} w_i \geq 1/n$. *If* $t \geq 1$, *then*

$$\sum_{\substack{T\subseteq N\\|T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}(T) = \binom{n}{t} \sum_{i=1}^{t} w_i.$$

Proof of Proposition 9. Let $k \in \{1, \ldots, n-1\}$. By Lemma 2 we have

$$\begin{aligned}
\operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}\right) &= 1 - \frac{1}{n-k} \sum_{t=1}^{n-k} \frac{1}{\binom{n}{t}} \sum_{\substack{T\subseteq N\\|T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}(T) = 1 - \frac{1}{n-k} \sum_{t=1}^{n-k} \sum_{i=1}^{t} w_i\\
&= 1 - \frac{1}{n-k} \sum_{i=1}^{n-k} (n+1-k-i) w_i = \operatorname{conj}_k(O_{\boldsymbol{w}}).
\end{aligned}$$

Analogously,

$$\operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right) = \frac{1}{n-k}\sum_{t=k}^{n-1}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}(T) = \frac{1}{n-k}\sum_{t=k}^{n-1}\sum_{i=1}^{t} w_i = \frac{1}{n-k}\sum_{t=k}^{n-1}\left(1-\sum_{i=t+1}^{n} w_i\right)$$

$$= \frac{1}{n-k}\left(n-k-\sum_{t=k}^{n-1}\sum_{i=t+1}^{n} w_i\right) = 1 - \frac{1}{n-k}\sum_{i=k+1}^{n}(i-k)w_i = \operatorname{disj}_k(O_{\boldsymbol{w}}). \qquad \square$$

Before giving the proof of Proposition 10, we previously establish the following lemma (the proof of this lemma can be found in Llamazares [22]).

Lemma 3. *Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$. If $t \geq 1$, then*

$$\sum_{\substack{T\subseteq N\\|T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) = \binom{n}{t}\sum_{i=1}^{t} w_i.$$

Proof of Proposition 10. Notice that under the hypothesis of Lemma 3 we have

$$\sum_{\substack{T\subseteq N\\|T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) = \binom{n}{t}\sum_{i=1}^{t} w_i.$$

Therefore, the proof is similar to that of Proposition 9 and hence is omitted here. □

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