On the relationship between the Crescent Method and SUOWA operators

Bonifacio Llamazares

***Abstract*—Different families of functions have been proposed in the literature with the purpose of simultaneously generalizing weighted means and OWA operators (see, for instance, WOWA and SUOWA operators). Recently, Jin, Mesiar, and Yager [L. Jin, R. Mesiar, and R. R. Yager, Melting probability measure with OWA operator to generate fuzzy measure: the Crescent Method, IEEE Transactions on Fuzzy Systems, in press] have introduced in the literature a new procedure called The Crescent Method to melt additive capacities with those of OWA operators. The main aim of this paper is to establish a relationship between the Crescent Method and SUOWA operators. From this relationship we give closed-form expressions for some well-known indices such as the Shapley values, the veto and favor indices, and the k-conjunctiveness and k-disjunctiveness indices.**

***Index Terms*—The Crescent Method, SUOWA operators, OWA operators, weighted means, Choquet integral.**

I. Introduction

Choquet integral [1] has become an important tool in some scientific fields due to its versatility, simplicity and good properties. Two of the best-known specific cases of Choquet integral are the weighted means and the ordered weighted averaging (OWA) operators [2]. Due to the importance of both families of functions, an interesting subject in this field is the construction of functions that allow us to simultaneously combine weighted means and OWA operators. With this purpose, several families of functions have been introduced in the literature (see, for instance, [3]–[6], and [7], [8] for an analysis of some of them). Among them, the weighted ordered weighted averaging (WOWA) operators [3] and the semiuninorm-based ordered weighted averaging (SUOWA) operators [4] stand out because they can be expressed through Choquet integrals with respect to known normalized capacities (a comparative analysis on the behavior of both families of functions has been carried out in [9]). It is worth noting that SUOWA operators have two important advantages over WOWA operators. On the one hand, for some specific cases of SUOWA operators it is possible to get closed-form expressions of some indices such as the orness degree, the Shapley value, the veto and favor indices, and the k-conjunctiveness and k-disjunctiveness indices (on the definition and properties of these indices see, for instance, [10]). On the other hand, some particular cases of SUOWA operators range between two order statistics (thus discarding extreme values). A summary of the main properties of SUOWA operators can be found in [11].

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The author is with the Departamento de Economía Aplicada and the Instituto de Matemáticas (IMUVA), Universidad de Valladolid, 47011 Valladolid, Spain, e-mail: boni@eco.uva.es.

Recently, a new procedure has been introduced in the literature to melt additive capacities with those of OWA operators: The Crescent Method [12]. In this method, the capacity is obtained as a convex combination (using for this the weighting vector $\boldsymbol{p}$) of n previously constructed capacities.

In this paper we establish a relationship between the Crescent Method and SUOWA operators, and we show that the capacity obtained with the Crescent Method can be expressed with one, or the dual of one, obtained in the context of SUOWA operators. This fact allows that some results already known for SUOWA operators can be applied to the Crescent Method. Specifically, we give some conditions on the weighting vectors that allow us to get capacities, and we show closed-form expressions of some indices (such as the Shapley value, the veto and favor indices, and the k-conjunctiveness and k-disjunctiveness indices) in those cases.

The remainder of the paper is organized as follows. In Section II we recall basic concepts on SUOWA operators. In Section III we remember the Crescent Method and establish the relationship between this method and SUOWA operators. Moreover, by using this relationship, we show closed-form expressions of some indices for the capacities obtained with the Crescent Method. Finally, some concluding remarks are provided in Section IV.

II. Basics on SUOWA operators

The following notation will be used throughout the paper: N denotes the set $\{1,\dots,n\}$, $|A|$ and A^{c} denote, respectively, the cardinality and the complement of a subset A of N, $\boldsymbol{\eta}$ is the tuple $(1/n,\dots,1/n) \in \mathbb{R}^n$, and, for each $k \in N$, $\boldsymbol{e}_k$ denotes the vector with 1 in the kth coordinate and 0 elsewhere. Vectors are denoted in bold, and given $\boldsymbol{x} \in \mathbb{R}^n$, $[\cdot]$ and $(\cdot)$ denote permutations such that $x_{[1]} \geq \cdots \geq x_{[n]}$ and $x_{(1)} \leq \cdots \leq x_{(n)}$.

SUOWA operators (and also the Crescent Method) are specific cases of Choquet integrals (on this subject, see, among others, [1], [10], [13]), which are constructed by using normalized capacities. A game υ on N is a set function, $\upsilon : 2^N \longrightarrow \mathbb{R}$ satisfying $\upsilon(\varnothing) = 0$. A monotonic game is called a capacity, and a capacity μ is normalized if $\mu(N) = 1$.

The monotonic cover (see [14], [15]) of a game υ is the set function $\hat{\upsilon}$ given by

$$\hat{\upsilon}(A) = \max_{B \subseteq A} \upsilon(B).$$

By construction $\hat{\upsilon}$ is a capacity on N, and $\hat{\upsilon} = \upsilon$ when υ is a capacity. Moreover, $\hat{\upsilon}$ is a normalized capacity when $\upsilon(N) = 1$ and $\upsilon(A) \leq 1$ for all $A \subseteq N$.

The Choquet integral with respect to a normalized capacity μ is the function $\mathcal{C}_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu(A_{[i]})\big(x_{[i]} - x_{[i+1]}\big),$$

where $A_{[i]} = \{[1], \dots, [i]\}$, and we adopt the convention that $x_{[n+1]} = 0$. Alternatively, the Choquet integral can be also expressed as

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \big(\mu(A_{[i]}) - \mu(A_{[i-1]})\big) x_{[i]},$$

where we use the convention $A_{[0]} = \varnothing$.

The best-known specific cases of Choquet integral are the weighted means and the OWA operators, which are defined by using weighting vectors.[1] The weighted mean $M_{\boldsymbol{p}}$ associated with a weighting vector $\boldsymbol{p}$ is the Choquet integral with respect to the normalized capacity $\mu_{\boldsymbol{p}}(A) = \sum_{i \in A} p_i$; that is,

$$M_{\boldsymbol{p}}(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i.$$

For its part, the OWA operator $O_{\boldsymbol{w}}$ associated with a weighting vector $\boldsymbol{w}$ is the Choquet integral with respect to the normalized capacity $\mu_{|\boldsymbol{w}|}(A) = \sum_{i=1}^{|A|} w_i$; that is,

$$O_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{[i]}.$$

Specific cases of OWA operators are the order statistics: the kth order statistic $\mathrm{OS}_k(\boldsymbol{x}) = x_{(k)}$ is the OWA operator associated with the vector $\boldsymbol{e}_{n-k+1}$; equivalently, $O_{\boldsymbol{e}_k} = \mathrm{OS}_{n-k+1}$. Nondecreasing ($w_1 \le \dots \le w_n$) and nonincreasing ($w_1 \ge \dots \ge w_n$) weighting vectors generate some well-known families of OWA operators [16].

The dual of a game υ is the game defined by

$$\overline{\upsilon}(A) = 1 - \upsilon(A^{\mathrm{c}}) \qquad (A \subseteq N).$$

Notice that the dual of a normalized capacity is also a normalized capacity. In the case of OWA operators, the dual of $\mu_{|\boldsymbol{w}|}$, $\overline{\mu}_{|\boldsymbol{w}|}$, is given by $\mu_{|\overline{\boldsymbol{w}}|}$, where $\overline{\boldsymbol{w}}$ is the dual of $\boldsymbol{w}$; that is, $\overline{\boldsymbol{w}} = (w_n, w_{n-1}, \dots, w_1)$ (equivalently, $\overline{w}_i = w_{n+1-i}$).

The concept of orness was proposed by Yager in the analysis of OWA operators [2] and it was generalized by Marichal to the case of Choquet integrals [17]. If μ is a normalized capacity on N, the orness degree of μ is

$$\mathrm{orness}(\mu) = \frac{1}{n-1} \sum_{t=1}^{n-1} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T| = t}} \mu(T).$$

In the case of capacities associated with OWA operators, we have

$$\mathrm{orness}(\mu_{|\boldsymbol{w}|}) = \frac{1}{n-1} \sum_{i=1}^{n} (n-i) w_i.$$

Semiuninorms [18] play a fundamental role in the definition of SUOWA operators. A semiuninorm is a nondecreasing binary operation $U : [0,1]^2 \longrightarrow [0,1]$ that has a neutral element. The semiuninorms used in the definition of SUOWA operators have $1/n$ as neutral element and they have to belong to the following subset (see [4]):

$$\widetilde{\mathcal{U}}^{1/n} = \left\{ U \in \mathcal{U}^{1/n} \mid U(1/k, 1/k) \le 1/k \text{ for all } k \in N \right\},$$

where $\mathcal{U}^{1/n}$ denotes the set of semiuninorms with neutral element $1/n$.

SUOWA operators were introduced in [4] and, since then, several papers have been published concerning their properties. For instance, in [19] it is shown that when $n = 2$ any Choquet integral with respect to a normalized capacity can be written as a SUOWA operator (see [20] for a graphical interpretation of the Choquet integral when $n = 2$). In [21] several continuous semiuninorms were introduced by using ordinal sums of aggregation operators (on this topic, see [22]). In [23], closed-form expressions of the orness degree [17], the Shapley value [24], [25], and the veto and favor indices [17], [25] were given for some specific cases of SUOWA operators. In [16] some SUOWA operators are generated by using unimodal weighting vectors (on unimodal sequences of real numbers see, for instance, [26]). The analysis of the conjunctive/disjunctive character of some specific cases of SUOWA operators has been carried out in [27]. Finally, a generalization of the Winsorized means [28], [29] through SUOWA operators has been introduced in [30].

Definition 1: Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let $U \in \widetilde{\mathcal{U}}^{1/n}$.

1) The game associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U is the set function $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}} : 2^N \longrightarrow \mathbb{R}$ defined by
$$\begin{aligned}\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) &= |A|\, U\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, \frac{\mu_{|\boldsymbol{w}|}(A)}{|A|}\right) \\ &= |A|\, U\left(\frac{\sum_{i \in A} p_i}{|A|}, \frac{\sum_{i=1}^{|A|} w_i}{|A|}\right),\end{aligned}$$
if $A \ne \varnothing$, and $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(\varnothing) = 0$.
2) $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$, the monotonic cover of the game $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}$, will be called the capacity associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U.
3) The SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U is the Choquet integral with respect to the capacity $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$.

By construction, SUOWA operators allow us to recover the weighted mean when $\boldsymbol{w} = \boldsymbol{\eta}$ and the OWA operator when $\boldsymbol{p} = \boldsymbol{\eta}$; that is, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{\eta}} = \mu_{\boldsymbol{p}}$ and $\hat{\upsilon}^U_{\boldsymbol{\eta},\boldsymbol{w}} = \mu_{|\boldsymbol{w}|}$ for any $U \in \widetilde{\mathcal{U}}^{1/n}$.

A continuous semiuninorm of special interest for its relationship with the Crescent Method is the following [21]:

$$U_{\widetilde{P}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in (1/n, 1]^2, \\ nxy & \text{otherwise.} \end{cases}$$

Given a nonempty subset A of N, we get

$$\upsilon^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}(A) = \frac{n}{|A|} \left(\sum_{i \in A} p_i\right) \left(\sum_{i=1}^{|A|} w_i\right), \tag{1}$$

whenever $\mu_{|\boldsymbol{w}|}(A) = \sum_{i=1}^{|A|} w_i \le |A|/n$. Notice that expression (1) is also valid for any semiuninorm $U \in \widetilde{\mathcal{U}}^{1/n}$ such that

[1] A weighting vector is a vector $\boldsymbol{q} \in [0,1]^n$ such that $\sum_{i=1}^{n} q_i = 1$.

$U(x,y) = nxy$ for all $y \leq 1/n$; that is, if

$$\widetilde{\mathcal{U}}_P^{1/n} = \left\{ U \in \widetilde{\mathcal{U}}^{1/n} \mid U(x,y) = nxy \text{ for all } y \leq 1/n \right\},$$

and $U \in \widetilde{\mathcal{U}}_P^{1/n}$, then

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}(A) = \frac{n}{|A|} \left(\sum_{i\in A} p_i \right) \left(\sum_{i=1}^{|A|} w_i \right) \tag{2}$$

for any nonempty subset A of N such that $\sum_{i=1}^{|A|} w_i \leq |A|/n$.

Obviously, when $\boldsymbol{w}$ satisfies that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$ the game $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\tilde{P}}}$ is given by

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\tilde{P}}}(A) = \frac{n}{|A|} \left(\sum_{i\in A} p_i \right) \left(\sum_{i=1}^{|A|} w_i \right),$$

whenever $A \neq \varnothing$, but it does not have to be a capacity [23]. However, we can guarantee that it is a capacity when $\boldsymbol{w}$ is a nondecreasing weighting vector [23] or satisfies the condition given in Proposition 2, which has been given under a different formulation in [11, Proposition 21]. To see the equivalence between both formulations, we establish the following result.

Lemma 1: Let $\boldsymbol{w}$ be a weighting vector. Then:

1) The sequence formed with the means of $\boldsymbol{w}$, $\left(\frac{1}{j}\sum_{i=1}^{j} w_i\right)_{j=1}^{n}$, is nondecreasing if and only if $\frac{1}{j}\sum_{i=1}^{j} w_i \leq w_{j+1}$ for all $j \in \{1,\ldots,n-1\}$.
2) If $\frac{1}{j}\sum_{i=1}^{j} w_i \leq w_{j+1}$ for all $j \in \{1,\ldots,n-1\}$, then $\frac{1}{j}\sum_{i=1}^{j} w_i \leq \frac{1}{n}$ for all $j \in \{1,\ldots,n-1\}$.

Proof: Let $\boldsymbol{w}$ be a weighting vector.

1) For each $j \in \{1,\ldots,n-1\}$ we have

$$\frac{1}{j}\sum_{i=1}^{j} w_i \leq \frac{1}{j+1}\sum_{i=1}^{j+1} w_i \;\Leftrightarrow\; (j+1)\sum_{i=1}^{j} w_i \leq j\sum_{i=1}^{j+1} w_i$$
$$\Leftrightarrow\; \sum_{i=1}^{j} w_i \leq j w_{j+1}.$$

2) It is obvious taking into account that, by the first item, the sequence $\left(\frac{1}{j}\sum_{i=1}^{j} w_i\right)_{j=1}^{n}$ is nondecreasing and its last component is $1/n$.

■

Proposition 1: Let $\boldsymbol{w}$ be a nondecreasing weighting vector. Then, for any weighting vector $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\tilde{P}}}$ is a normalized capacity on N given by

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\tilde{P}}}(A) = \frac{n}{|A|} \left(\sum_{i\in A} p_i \right) \left(\sum_{i=1}^{|A|} w_i \right),$$

whenever $A \neq \varnothing$.

Proposition 2: Let $\boldsymbol{w}$ be a weighting vector such that the sequence $\left(\frac{1}{j}\sum_{i=1}^{j} w_i\right)_{j=1}^{n}$ is nondecreasing. Then, for any weighting vector $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\tilde{P}}}$ is a normalized capacity on N given by

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\tilde{P}}}(A) = \frac{n}{|A|} \left(\sum_{i\in A} p_i \right) \left(\sum_{i=1}^{|A|} w_i \right),$$

whenever $A \neq \varnothing$.

Note that any nondecreasing weighting vector also satisfies the condition given in Proposition 2. Therefore, Proposition 2 allows us to expand the set of weighting vectors for which we get capacities. For instance, the weighting vector $\boldsymbol{w} = (0.1, 0.3, 0.2, 0.4)$ is not nondecreasing but $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\tilde{P}}}$ is a normalized capacity on N for any weighting vector $\boldsymbol{p}$. Notice again that the above results are also valid for any semiuninorm $U \in \widetilde{\mathcal{U}}_P^{1/n}$.

III. The Crescent Method and its relationship to SUOWA operators

The Crescent Method [12] has recently been introduced in the literature with the aim of melting additive capacities with those of OWA operators. Given two weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$, the Crescent Method can be implemented through the following steps:

Step 1: Generate the Crescent extent $c'_{\boldsymbol{w}} = \left(c'_{\boldsymbol{w}}(i)\right)_{i=1}^{n-1}$ and the Dual Crescent extent $d'_{\boldsymbol{w}} = \left(d'_{\boldsymbol{w}}(i)\right)_{i=1}^{n-1}$ given by

$$c'_{\boldsymbol{w}}(i) = \frac{\min\left(\sum_{k=1}^{i} w_k, \frac{i}{n}\right)}{\frac{i}{n}},$$

$$d'_{\boldsymbol{w}}(i) = \frac{1 - \max\left(\sum_{k=1}^{i} w_k, \frac{i}{n}\right)}{1 - \frac{i}{n}}.$$

Step 2: For each $j \in N$, construct the game $\mu_j^{\boldsymbol{w}}$ defined by

$$\mu_j^{\boldsymbol{w}}(A) = \begin{cases} c'_{\boldsymbol{w}}(|A|) & \text{if } j \in A, \\ 1 - d'_{\boldsymbol{w}}(|A|) & \text{if } j \notin A, \end{cases}$$

for any $\varnothing \subsetneq A \subsetneq N$, $\mu_j^{\boldsymbol{w}}(\varnothing) = 0$, and $\mu_j^{\boldsymbol{w}}(N) = 1$.

Step 3: Construct the game $\mu^{\boldsymbol{p},\boldsymbol{w}}$ given by

$$\mu^{\boldsymbol{p},\boldsymbol{w}} = \sum_{j=1}^{n} p_j \mu_j^{\boldsymbol{w}}.$$

The Crescent Method has the following properties [12].

Proposition 3: Let $\boldsymbol{w}$ be a nondecreasing or a nonincreasing weighting vector, or $\boldsymbol{w} = \boldsymbol{e}_k$ for some $k \in N$. Then:

1) For each $j \in N$, $\mu_j^{\boldsymbol{w}}$ is a normalized capacity and, consequently, $\mu^{\boldsymbol{p},\boldsymbol{w}}$ is also a normalized capacity.
2) If $\boldsymbol{w} = \boldsymbol{\eta}$, then $\mu^{\boldsymbol{p},\boldsymbol{\eta}} = \mu_{\boldsymbol{p}}$.
3) If $\boldsymbol{p} = \boldsymbol{\eta}$, then $\mu^{\boldsymbol{\eta},\boldsymbol{w}} = \mu_{|\boldsymbol{w}|}$.
4) $\operatorname{orness}(\mu^{\boldsymbol{p},\boldsymbol{w}}) = \operatorname{orness}(\mu_{|\boldsymbol{w}|})$ for any weighting vector $\boldsymbol{p}$.
5) If $\boldsymbol{w}$ is a nondecreasing weighting vector, then $\mu^{\boldsymbol{p},\boldsymbol{w}} \leq \mu_{\boldsymbol{p}}$ for any weighting vector $\boldsymbol{p}$.[2]
6) If $\boldsymbol{w}$ is a nonincreasing weighting vector, then $\mu^{\boldsymbol{p},\boldsymbol{w}} \geq \mu_{\boldsymbol{p}}$ for any weighting vector $\boldsymbol{p}$.[3]

[2]Note that the result is also valid when $\boldsymbol{w}$ satisfies that $\sum_{k=1}^{i} w_k \leq i/n$ for all $i \in \{1,\ldots,n-1\}$, but in this case we are not guaranteed that $\mu^{\boldsymbol{p},\boldsymbol{w}}$ is a capacity (see the comments after expression (4)).

[3]A remark similar to the previous one can also be made in this case.

Next we are going to give explicitly the values that the game $\mu^{\boldsymbol{p},\boldsymbol{w}}$ takes.

Proposition 4: Let $\boldsymbol{w}$ and $\boldsymbol{p}$ be two weighting vectors. If $\varnothing \subsetneq A \subsetneq N$, then

$$\mu^{\boldsymbol{p},\boldsymbol{w}}(A) = \begin{cases} \dfrac{n}{|A|}\displaystyle\sum_{j\in A} p_j \sum_{k=1}^{|A|} w_k & \text{if } \displaystyle\sum_{k=1}^{|A|} w_k \le \frac{|A|}{n}, \\ 1-\dfrac{n}{n-|A|}\displaystyle\sum_{j\notin A} p_j \sum_{k=|A|+1}^{n} w_k & \text{if } \displaystyle\sum_{k=1}^{|A|} w_k > \frac{|A|}{n}, \end{cases}$$

or, equivalently,

$$\mu^{\boldsymbol{p},\boldsymbol{w}}(A) = \begin{cases} \dfrac{n}{|A|}\mu_p(A)\mu_{|\boldsymbol{w}|}(A) & \text{if } \mu_{|\boldsymbol{w}|}(A) \le \dfrac{|A|}{n}, \\ \overline{\mu}^{\boldsymbol{p},\overline{\boldsymbol{w}}}(A) & \text{if } \mu_{|\boldsymbol{w}|}(A) > \dfrac{|A|}{n}. \end{cases}$$

Proof: If $\varnothing \subsetneq A \subsetneq N$, then

$$\begin{aligned}\mu^{\boldsymbol{p},\boldsymbol{w}}(A) &= \sum_{j\in A} p_j \mu_j^{\boldsymbol{w}}(A) + \sum_{j\notin A} p_j \mu_j^{\boldsymbol{w}}(A) \\ &= c'_{\boldsymbol{w}}(|A|)\sum_{j\in A} p_j + \big(1-d'_{\boldsymbol{w}}(|A|)\big)\sum_{j\notin A} p_j.\end{aligned}$$

We distinguish two cases:

1) If $\sum_{k=1}^{|A|} w_k \le |A|/n$, then

$$c'_{\boldsymbol{w}}(|A|) = \frac{n}{|A|}\sum_{k=1}^{|A|} w_k, \quad \text{and} \quad 1-d'_{\boldsymbol{w}}(|A|) = 0.$$

Therefore,

$$\mu^{\boldsymbol{p},\boldsymbol{w}}(A) = \frac{n}{|A|}\sum_{j\in A} p_j \sum_{k=1}^{|A|} w_k.$$

2) If $\sum_{k=1}^{|A|} w_k > |A|/n$, then

$$c'_{\boldsymbol{w}}(|A|) = 1,$$

and

$$1-d'_{\boldsymbol{w}}(|A|) = 1 - \frac{n}{n-|A|}\sum_{k=|A|+1}^{n} w_k.$$

Therefore,

$$\begin{aligned}\mu^{\boldsymbol{p},\boldsymbol{w}}(A) &= \sum_{j\in A} p_j + \left(1 - \frac{n}{n-|A|}\sum_{k=|A|+1}^{n} w_k\right)\sum_{j\notin A} p_j \\ &= 1 - \frac{n}{n-|A|}\sum_{j\notin A} p_j \sum_{k=|A|+1}^{n} w_k.\end{aligned}$$

Lastly, notice that

$$\sum_{k=1}^{|A|} w_k > \frac{|A|}{n} \quad \Leftrightarrow \quad \sum_{k=|A|+1}^{n} w_k < \frac{n-|A|}{n},$$

and, in this case

$$1 - \frac{n}{n-|A|}\sum_{j\notin A} p_j \sum_{k=|A|+1}^{n} w_k = 1 - \mu^{\boldsymbol{p},\overline{\boldsymbol{w}}}(A^{\mathrm{c}}) = \overline{\mu}^{\boldsymbol{p},\overline{\boldsymbol{w}}}(A).$$

■

According to Proposition 4 and the results shown in Section II, the value $\mu^{\boldsymbol{p},\boldsymbol{w}}(A)$ can be expressed by means of $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A)$ (where $U \in \widetilde{\mathcal{U}}_P^{1/n}$) when $\mu_{|\boldsymbol{w}|}(A) \le |A|/n$ and through the dual of $\upsilon^U_{\boldsymbol{p},\overline{\boldsymbol{w}}}(A)$ when $\mu_{|\boldsymbol{w}|}(A) > |A|/n$; that is,

$$\mu^{\boldsymbol{p},\boldsymbol{w}}(A) = \begin{cases} \upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) & \text{if } \mu_{|\boldsymbol{w}|}(A) \le \dfrac{|A|}{n}, \\ \overline{\upsilon}^U_{\boldsymbol{p},\overline{\boldsymbol{w}}}(A) & \text{if } \mu_{|\boldsymbol{w}|}(A) > \dfrac{|A|}{n}. \end{cases} \tag{3}$$

Notice that when $\mu_{|\boldsymbol{w}|}(A) = |A|/n$ then $\mu^{\boldsymbol{p},\boldsymbol{w}}(A) = \sum_{j\in A} p_j = \overline{\upsilon}^U_{\boldsymbol{p},\overline{\boldsymbol{w}}}(A)$. So, $\mu^{\boldsymbol{p},\boldsymbol{w}}$ can also be expressed as

$$\mu^{\boldsymbol{p},\boldsymbol{w}}(A) = \begin{cases} \upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) & \text{if } \mu_{|\boldsymbol{w}|}(A) < \dfrac{|A|}{n}, \\ \overline{\upsilon}^U_{\boldsymbol{p},\overline{\boldsymbol{w}}}(A) & \text{if } \mu_{|\boldsymbol{w}|}(A) \ge \dfrac{|A|}{n}. \end{cases} \tag{4}$$

It is important to point out that $\mu^{\boldsymbol{p},\boldsymbol{w}}$ is a game but it is not, in general, a capacity. For instance, consider the weighting vectors $\boldsymbol{p} = (0.1, 0.2, 0.3, 0.4)$ and $\boldsymbol{w} = (0.2, 0.1, 0.3, 0.4)$.[4] Then,

$$\mu^{\boldsymbol{p},\boldsymbol{w}}(\{4\}) = 0.32 \;>\; 0.3 = \mu^{\boldsymbol{p},\boldsymbol{w}}(\{1,4\}).$$

From [12] we know some families of weighting vectors for which $\mu^{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity. Expression (3) is very relevant because it allows us to expand these sets of weighting vectors. For instance, expression (3) together with Proposition 2 allow us to establish the following result.

Corollary 1: Let $\boldsymbol{w}$ be a weighting vector such that the sequence $\left(\frac{1}{j}\sum_{i=1}^{j} w_i\right)_{j=1}^{n}$ is nondecreasing. Then, for any weighting vector $\boldsymbol{p}$, $\mu^{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity on N given by

$$\mu^{\boldsymbol{p},\boldsymbol{w}}(A) = \upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) = \frac{n}{|A|}\left(\sum_{i\in A} p_i\right)\left(\sum_{i=1}^{|A|} w_i\right),$$

whenever $A \neq \varnothing$ and $U \in \widetilde{\mathcal{U}}_P^{1/n}$.

An analogous result to Corollary 1 can be obtained when the sequence $\left(\frac{1}{n-j+1}\sum_{i=j}^{n} w_i\right)_{j=1}^{n}$ is nonincreasing.[5]

Proposition 5: Let $\boldsymbol{w}$ be a weighting vector such that the sequence $\left(\frac{1}{n-j+1}\sum_{i=j}^{n} w_i\right)_{j=1}^{n}$ is nonincreasing. Then, for any weighting vector $\boldsymbol{p}$, $\mu^{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity given by

$$\mu^{\boldsymbol{p},\boldsymbol{w}}(A) = \overline{\upsilon}^U_{\boldsymbol{p},\overline{\boldsymbol{w}}}(A) = 1 - \frac{n}{n-|A|}\sum_{j\notin A} p_j \sum_{k=|A|+1}^{n} w_k,$$

whenever $\varnothing \subsetneq A \subsetneq N$ and $U \in \widetilde{\mathcal{U}}_P^{1/n}$.

Proof: Let $\boldsymbol{w}$ be a weighting vector such that the sequence $\left(\frac{1}{n-j+1}\sum_{i=j}^{n} w_i\right)_{j=1}^{n}$ is nonincreasing. Since the first element of the sequence is $1/n$, we have $1/n \ge$

[4] Note the similarity of these weighting vectors with those used in Example 4 in [12].

[5] Notice that any nonincreasing weighting vector satisfies this condition.

$\left(\sum_{i=j+1}^{n} w_i\right)/(n-j)$ for any $j \in \{1,\dots,n-1\}$. But

$$\begin{aligned}\frac{1}{n} \geq \frac{1}{n-j}\sum_{i=j+1}^{n} w_i \;&\Leftrightarrow\; \sum_{i=j+1}^{n} w_i \leq \frac{n-j}{n}\\ &\Leftrightarrow\; 1-\sum_{i=j+1}^{n} w_i \geq 1-\frac{n-j}{n}\\ &\Leftrightarrow\; \sum_{i=1}^{j} w_i \geq \frac{j}{n}.\end{aligned}$$

Hence, $\boldsymbol{w}$ satisfies $\mu_{|\boldsymbol{w}|}(A) \geq |A|/n$ for any $\varnothing \subsetneq A \subsetneq N$ and, by expression (4),

$$\mu^{\boldsymbol{p},\boldsymbol{w}}(A) = \overline{\upsilon}^{U}_{\boldsymbol{p},\overline{\boldsymbol{w}}}(A) = 1 - \frac{n}{n-|A|}\sum_{j\notin A} p_j \sum_{k=|A|+1}^{n} w_k.$$

Lastly, note that when the sequence $\left(\frac{1}{n-j+1}\sum_{i=j}^{n} w_i\right)_{j=1}^{n}$ is nonincreasing, the sequence $\left(\frac{1}{j}\sum_{i=1}^{j} w_{n+1-i}\right)_{j=1}^{n} = \left(\frac{1}{j}\sum_{i=1}^{j} \overline{w}_i\right)_{j=1}^{n}$ is nondecreasing and, by Corollary 1, $\upsilon^{U}_{\boldsymbol{p},\overline{\boldsymbol{w}}}$ is a normalized capacity; and so is its dual, $\overline{\upsilon}^{U}_{\boldsymbol{p},\overline{\boldsymbol{w}}}$. ■

In the case of weighting vectors associated with order statistics, we have the following result.

Corollary 2: If $\boldsymbol{w} = \boldsymbol{e}_k$ for some $k \in N$, then $\mu^{\boldsymbol{p},\boldsymbol{e}_k} = \mu_{|\boldsymbol{e}_k|}$ for any weighting vector $\boldsymbol{p}$.

Proof: The proof is immediate taking into account that if $\varnothing \subsetneq A \subsetneq N$, then

$$\begin{aligned}\mu^{\boldsymbol{p},\boldsymbol{e}_k}(A) &= \begin{cases} 0 & \text{if } |A| < k,\\ 1 & \text{if } |A| \geq k,\end{cases}\\ &= \mu_{|\boldsymbol{e}_k|}(A).\end{aligned}$$

■

Therefore, for any weighting vector $\boldsymbol{p}$, when $\boldsymbol{w} = \boldsymbol{e}_k$ for some $k \in N$, the Choquet integral with respect to the capacity $\mu^{\boldsymbol{p},\boldsymbol{e}_k}$ is the OWA operator associated with the vector $\boldsymbol{e}_k$; that is, the order statistic OS_{n-k+1}.

In addition to the previous results, Corollary 1 and Proposition 5 also allow us to obtain closed-form expression of some indices such as the Shapley value [24], the veto and favor indices [17], and the k-conjunctiveness and k-disjunctiveness indices [25]. First we are going to define these indices. Given $j \in N$ and μ a normalized capacity on N, the Shapley value, the veto, and the favor indices of j with respect to μ are defined by

$$\phi(\mu,j) = \frac{1}{n}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\Big(\mu(T\cup\{j\})-\mu(T)\Big),$$

$$\mathrm{veto}(\mu,j) = 1-\frac{1}{n-1}\sum_{t=1}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\mu(T),$$

$$\mathrm{favor}(\mu,j) = \frac{1}{n-1}\sum_{t=0}^{n-2}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\mu(T\cup\{j\}).$$

Given $k \in N\setminus\{n\}$ and μ a normalized capacity on N, the k-conjunctiveness and k-disjunctiveness indices with respect to μ are defined by

$$\mathrm{conj}_k(\mu) = 1-\frac{1}{n-k}\sum_{t=1}^{n-k}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\mu(T),$$

$$\mathrm{disj}_k(\mu) = \frac{1}{n-k}\sum_{t=k}^{n-1}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\mu(T).$$

It is well known that the Shapley value is self-dual; that is, $\phi(\overline{\mu},j) = \phi(\mu,j)$ for all $j \in N$; and that $\mathrm{orness}(\overline{\mu}) = 1-\mathrm{orness}(\mu)$. Moreover, we also have (see [10], [17])

$$\begin{aligned}\mathrm{veto}(\overline{\mu},j) &= \mathrm{favor}(\mu,j), & \mathrm{conj}_k(\overline{\mu}) &= \mathrm{disj}_k(\mu),\\ \mathrm{favor}(\overline{\mu},j) &= \mathrm{veto}(\mu,j), & \mathrm{disj}_k(\overline{\mu}) &= \mathrm{conj}_k(\mu).\end{aligned}$$

Proposition 6: Let $\boldsymbol{w}$ be a weighting vector such that the sequence $\left(\frac{1}{j}\sum_{i=1}^{j} w_i\right)_{j=1}^{n}$ is nondecreasing. Then, for any weighting vector $\boldsymbol{p}$, any $j \in N$, and any $k \in N\setminus\{n\}$, we have

$$\begin{aligned}\phi\big(\mu^{\boldsymbol{p},\boldsymbol{w}},j\big) &= \frac{1-p_j+(np_j-1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right)w_i}{n-1},\\ \mathrm{veto}\left(\mu^{\boldsymbol{p},\boldsymbol{w}},j\right) &= 1-\frac{n}{n-1}(1-p_j)\,\mathrm{orness}(\mu_{|\boldsymbol{w}|}),\\ \mathrm{favor}\left(\mu^{\boldsymbol{p},\boldsymbol{w}},j\right) &= 1-\mathrm{veto}\left(\mu^{\boldsymbol{p},\boldsymbol{w}},j\right)+\frac{n\phi\big(\mu^{\boldsymbol{p},\boldsymbol{w}},j\big)-1}{n-1},\\ \mathrm{conj}_k\left(\mu^{\boldsymbol{p},\boldsymbol{w}}\right) &= \mathrm{conj}_k(\mu_{|\boldsymbol{w}|}),\\ \mathrm{disj}_k\left(\mu^{\boldsymbol{p},\boldsymbol{w}}\right) &= \mathrm{disj}_k(\mu_{|\boldsymbol{w}|}).\end{aligned}$$

Proof: It is immediate taking into account Corollary 1 and the results given in [23], [27]. ■

Proposition 7: Let $\boldsymbol{w}$ be a weighting vector such that the sequence $\left(\frac{1}{n-j+1}\sum_{i=j}^{n} w_i\right)_{j=1}^{n}$ is nonincreasing. Then, for any weighting vector $\boldsymbol{p}$, any $j \in N$, and any $k \in N\setminus\{n\}$, we have

$$\begin{aligned}\phi\big(\mu^{\boldsymbol{p},\boldsymbol{w}},j\big) &= \frac{1-p_j+(np_j-1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right)w_{n+1-i}}{n-1},\\ \mathrm{favor}\left(\mu^{\boldsymbol{p},\boldsymbol{w}},j\right) &= 1-\frac{n}{n-1}(1-p_j)\big(1-\mathrm{orness}(\mu_{|\boldsymbol{w}|})\big),\\ \mathrm{veto}\left(\mu^{\boldsymbol{p},\boldsymbol{w}},j\right) &= 1-\mathrm{favor}\left(\mu^{\boldsymbol{p},\boldsymbol{w}},j\right)+\frac{n\phi\big(\mu^{\boldsymbol{p},\boldsymbol{w}},j\big)-1}{n-1},\\ \mathrm{conj}_k\left(\mu^{\boldsymbol{p},\boldsymbol{w}}\right) &= \mathrm{conj}_k(\mu_{|\boldsymbol{w}|}),\\ \mathrm{disj}_k\left(\mu^{\boldsymbol{p},\boldsymbol{w}}\right) &= \mathrm{disj}_k(\mu_{|\boldsymbol{w}|}).\end{aligned}$$

Proof: The proofs are straightforward taking into account Propositions 5 and 6, and the relationships given before Proposition 6. For instance, given $U \in \widetilde{\mathcal{U}}_P^{1/n}$,

$$\phi\big(\mu^{\boldsymbol{p},\boldsymbol{w}},j\big) = \phi\big(\overline{\upsilon}^{U}_{\boldsymbol{p},\overline{\boldsymbol{w}}},j\big) = \phi\big(\upsilon^{U}_{\boldsymbol{p},\overline{\boldsymbol{w}}},j\big)$$

$$= \frac{1 - p_j + (np_j - 1) \sum_{i=1}^{n} \left(\sum_{t=i}^{n} \frac{1}{t} \right) w_{n+1-i}}{n-1}.$$

The expressions for the remaining indices can be obtained in a similar way. ∎

IV. Conclusion

The Crescent Method has recently been introduced in the literature with the intent of melting additive capacities with those of OWA operators. In this paper we have established a relationship between the Crescent Method and the SUOWA operators. Specifically, we have shown that the capacity obtained with the Crescent Method can be expressed with one, or the dual of one, obtained in the context of SUOWA operators. This fact has allowed us to give closed-form expressions of some well-known indices, which is essential to have a better knowledge of these capacities.

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Bonifacio Llamazares received the M.Sc. degree in mathematics and the Ph.D. degree in economics from the University of Valladolid, Valladolid, Spain, in 1991 and 2000, respectively.

He is currently an Associate Professor with the Department of Applied Economics, University of Valladolid. He has authored/coauthored papers in several international journals including *Computers & Industrial Engineering*, *European Journal of Operational Research*, *Fuzzy Sets and Systems*, *Group Decision and Negotiation*, *Information Fusion*, *Information Sciences*, and *International Journal of Intelligence Systems*. His domain of research ranges from operations research to fuzzy logic, with special emphasis in the following topics: decision support systems, multicriteria decision making, games, aggregation operators, and fuzzy sets.

Dr. Llamazares is an Academic Editor for *Mathematical Problems in Engineering* and a reviewer for numerous journals. He has participated in the scientific committee of several conferences.