Shapley values and tolerance indices of the operators obtained with the Crescent Method

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Abstract

Several operators have emerged in the framework of Choquet integral with the purpose of simultaneously generalizing weighted means and ordered weighted averaging (OWA) operators. However, on many occasions, not enough attention has been paid to whether the constructed operators behaved similarly to the weighted means and OWA operators that have been generalized. In this sense, it seems necessary that these new operators preserve the weights assigned to the information sources (which are established through the weighting vector associated with the weighted mean) and that they are able to rule out extreme values (which is an important characteristic of OWA operators).

In this paper we analyze a family of operators recently introduced in the literature through the Crescent Method. First, we introduce a broad class of weighting vectors that allow us to guarantee that the games generated with the Crescent Method are capacities. Next we analyze the conjunctive/disjuntive character of the Choquet integrals associated with these capacities and we also give closed-form expressions of some tolerance and importance indices such as k-conjunctiveness/disjunctiveness indices, the veto and favor indices, and the Shapley values. Finally, we give two examples to show the usefulness of the results obtained.

Keywords: The Crescent Method, Semi-SUOWA operators, SUOWA operators, Shapley values, Tolerance indices, Winsorized weighted means.

1. Introduction

Weighted means and ordered weighted averaging (OWA) operators [1] are the best-known particular cases of the Choquet integral. Both families are defined by means of weighting vectors but, in the case of the weighted means, the vector reflects the importance of the information sources while, in the case of the OWA operators, the vector expresses the importance of order statistics (notice that an OWA operator is a convex combination of order statistics).

There exist in the literature several constructions that simultaneously generalize the weighted means and the OWA operators in the framework of Choquet integral (see, for instance, [2–4] for an analysis of some of them). Most of

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them consider functions parametrized by two weighting vectors, $\boldsymbol{p}$ for the weighted mean and $\boldsymbol{w}$ for the OWA operator, so that we can recover the weighted mean when $\boldsymbol{w} = (1/n, \dots, 1/n)$ and the OWA operator when $\boldsymbol{p} = (1/n, \dots, 1/n)$. This approach has given rise to new families of functions such as the weighted ordered weighted averaging (WOWA) operators [5], the semiuninorm-based ordered weighted averaging (SUOWA) operators [6, 7], the functions obtained from the Crescent Method [8], and the Semi-SUOWA operators [9, 10].

An important aspect in building the aforementioned families is that they preserve the main characteristics of weighted means and OWA operators, so that, for instance, they are able to discard extreme values (this is the case, for instance, when trimmed or Winsorized means are considered as OWA operators) while each information source has the desired weight.

However, in some constructions proposed in the literature, the vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ do not play the desired role. For instance, suppose four information sources, $n = 4$, and the weighting vectors $\boldsymbol{p} = (0.5, 0.2, 0.2, 0.1)$ and $\boldsymbol{w} = (0, 0.5, 0.5, 0)$. With the choice of $\boldsymbol{p}$ we want that the first information source has the same importance as the other three together, while $\boldsymbol{w}$ is chosen for the purpose of discarding extreme values (in this case, the maximum and minimum values). Consider now WOWA operators [5], which are the best-known generalizations of weighted means and OWA operators. It is well known (see, for instance [11, 12]) that WOWA operators are Choquet integrals with respect to the normalized capacities $\mu^Q_{\boldsymbol{p},\boldsymbol{w}}(A) = Q\left(\sum_{i\in A} p_i\right)$, where Q is a quantifier generating the weighting vector $\boldsymbol{w}$.[1] In order to determine the importance of each information source, it is usual to employ an importance index (normally the Shapley value [13]). The Shapley values can be found using the *Kappalab R package* [14], obtaining in our case:[2]

$$\phi_1\left(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}\right) = 2/3, \qquad \phi_2\left(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}\right) = \phi_3\left(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}\right) = 2/15, \qquad \phi_4\left(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}\right) = 1/15.$$

As can be seen, the weight of the first information source is double the sum of the other three together, instead of equal, as desired. Concerning the use of $\boldsymbol{w}$ to discard the maximum and minimum values, notice that, for instance,

$$W^Q_{\boldsymbol{p},\boldsymbol{w}}(10, 6, 6, 6) = 8, \qquad W^Q_{\boldsymbol{p},\boldsymbol{w}}(0, 6, 6, 6) = 3,$$

that is, the maximum and minimum values are taken into account in the aggregation process.

Taking into account the above remarks, when using an operator it seems necessary to have a certain knowledge about its behavior. In this sense, the study of the Shapley values and some tolerance indices has already been carried out for some particular cases of SUOWA operators (see [15, 16]).

In a recent paper [8], the Crescent Method was proposed as a new methodology for obtaining games that simultaneously generalize those of the weighted means and OWA operators. Moreover, it has been shown that these games are capacities when the weighting vector $\boldsymbol{w}$ is unimodal [9] (see also [17]).

In this paper we introduce a family of vectors that generalizes that of the unimodal weighting vectors and that guarantees that the games obtained with the Crescent Method are capacities. Furthermore, we study these capacities

[1] The WOWA operator associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and Q will be denoted as $W^Q_{\boldsymbol{p},\boldsymbol{w}}$.

[2] We have considered that Q is obtained using a linear interpolation, which is the most usual method in the field of WOWA operators.

and give closed-form expressions for the Shapley values and some tolerance indices such as k-conjunctiveness and k-disjunctiveness indices, and the veto and favor indices.[3] Of special interest are the expressions obtained for the Shapley values because they allow us to construct functions that give the desired weight to the information sources.

The remainder of the paper is organized as follows. In Section 2 we present basic concepts on Choquet integrals, weighted means, OWA operators and several indices used in the study of Choquet integrals. In Section 3 we recall the Crescent Method by using games associated with SUOWA operators and we introduce a family of vectors that generalizes to that of the unimodal weighting vectors and that allows us to obtain capacities. In Section 4 we give the main results of the paper while in Section 5 we show some particular cases of special interest. Section 6 is devoted to illustrate the utility of these operators through two examples. Finally, some concluding remarks are provided in Section 7.

2. Preliminaries

The following notation will be used throughout the paper: N denotes the set $\{1,\dots,n\}$, and A^c and $|A|$ denote, respectively, the complement and the cardinality of a subset A of N. Vectors are denoted in bold, $\boldsymbol{\eta}$ is the tuple $(1/n,\dots,1/n) \in \mathbb{R}^n$, and, for each $k \in N$, $\boldsymbol{e}_k$ denotes the vector with 1 in the kth coordinate and 0 elsewhere. Given $\boldsymbol{x} \in \mathbb{R}^n$, $[\cdot]$ and $(\cdot)$ denote permutations such that $x_{[1]} \geqslant \cdots \geqslant x_{[n]}$ and $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$. For any $a \in \mathbb{R}$, $\lfloor a \rfloor$ and $\lceil a \rceil$ denote, respectively, the floor and the ceiling of a; i.e., the largest integer smaller than or equal to a, and the smallest integer larger than or equal to a.

2.1. Choquet integrals

Games and capacities play a central role in the framework of Choquet integrals. A game υ on N is a set function $\upsilon : 2^N \longrightarrow \mathbb{R}$ satisfying $\upsilon(\varnothing) = 0$. Monotonic games are called capacities, and a capacity μ is normalized when $\mu(N) = 1$.

One direct way to construct a capacity from a game is through its monotonic cover (see [18, 19]). The monotonic cover of a game υ is the set function $\hat{\upsilon}$ defined by

$$\hat{\upsilon}(A) = \max_{B \subseteq A} \upsilon(B).$$

It is easy to check that $\hat{\upsilon}$ is a capacity and, when υ is a capacity, $\hat{\upsilon} = \upsilon$. Moreover, $\hat{\upsilon}$ is a normalized capacity when $\upsilon(A) \leqslant 1$ for all $A \subseteq N$ and there exists $B \subseteq N$ with $\upsilon(B) = 1$.

The Choquet integral (see, for instance, [20–22]) with respect to a normalized capacity μ is the function $C_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu(A_{[i]})\big(x_{[i]} - x_{[i+1]}\big),$$

[3] It is known that the orness degree of the functions obtained with the Crescent Method coincides with the orness degree of the OWA operator; see [8].

where $A_{[i]} = \{[1], \ldots, [i]\}$, and we use the convention $x_{[n+1]} = 0$. Alternatively, the Choquet integral can be also expressed as

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \big(\mu(A_{[i]}) - \mu(A_{[i-1]})\big) x_{[i]},$$

where we adopt the convention that $A_{[0]} = \varnothing$.

Two particular well-known cases of Choquet integral are the weighted means and the OWA operators, which are defined by using weighting vectors. A weighting vector is a vector $\boldsymbol{q} \in [0,1]^n$ such that $\sum_{i=1}^n q_i = 1$; and the set of weighting vectors will be denoted by $\mathcal{W}$.

The weighted mean $M_{\boldsymbol{p}}$ associated with a weighting vector $\boldsymbol{p}$ is the function

$$M_{\boldsymbol{p}}(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i,$$

which is a Choquet integral with respect to the normalized capacity $\mu_{\boldsymbol{p}}(A) = \sum_{i \in A} p_i$. Analogously, the OWA operator $O_{\boldsymbol{w}}$ associated with a weighting vector $\boldsymbol{w}$ is the function

$$O_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{[i]},$$

which is a Choquet integral with respect to the normalized capacity $\mu_{|\boldsymbol{w}|}(A) = \sum_{i=1}^{|A|} w_i$. Specific cases of OWA operators are the order statistics. The kth order statistic OS_k, which is defined by $\mathrm{OS}_k(\boldsymbol{x}) = x_{(k)}$, is the OWA operator associated with the vector $\boldsymbol{e}_{n+1-k}$; or, equivalently, $O_{\boldsymbol{e}_k} = \mathrm{OS}_{n+1-k}$.

The dual of a game υ is the game $\overline{\upsilon}$ defined by

$$\overline{\upsilon}(A) = \upsilon(N) - \upsilon(A^{\mathrm{c}}) \qquad (A \subseteq N).$$

It is worth noting that if υ is a normalized capacity, then $\overline{\upsilon}$ is also a normalized capacity given by

$$\overline{\upsilon}(A) = 1 - \upsilon(A^{\mathrm{c}}) \qquad (A \subseteq N).$$

In the case of OWA operators, the dual of $\mu_{|\boldsymbol{w}|}$, $\overline{\mu}_{|\boldsymbol{w}|}$, is given by $\mu_{|\overline{\boldsymbol{w}}|}$, where $\overline{\boldsymbol{w}}$ is the dual of $\boldsymbol{w}$; that is, $\overline{\boldsymbol{w}} = (w_n, w_{n-1}, \ldots, w_1)$ (equivalently, $\overline{w}_i = w_{n+1-i}$).

A prominent family of weighting vectors are the unimodal ones [23]. A weighting vector $\boldsymbol{w}$ is unimodal if there exists an index k such that

$$w_1 \leqslant \cdots \leqslant w_{k-1} \leqslant w_k \geqslant w_{k+1} \geqslant \cdots \geqslant w_n.$$

Notice that unimodal weighting vectors include, among others, nondecreasing ($w_1 \leqslant \cdots \leqslant w_n$), nonincreasing ($w_1 \geqslant \cdots \geqslant w_n$), and centered weighting vectors [24].

2.2. Indices for Choquet integrals

Several indices have been introduced in the literature in order to knowing the behavior of the functions used in the aggregation processes. Among the wide variety of indices that exist, it is worth noting the orness degree, the Shapley values and the tolerance indices. Although in this paper we will focus on Choquet integrals, it should be noted that some of these indices can be defined in a more general context.

The orness degree can be seen as a measure of the closeness of a Choquet integral to the maximum; or equivalently, a measure of the disjunctive character of the operator. It was proposed by Yager in the study of OWA operators [1] and, later, generalized by Marichal to the field of Choquet integrals [25].

Definition 1. Let μ be a normalized capacity on N. The orness degree of C_μ is defined by[4]

$$\text{orness}(\mu) = \frac{1}{n-1}\sum_{t=1}^{n-1}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\mu(T).$$

The Shapley value [13, 26] was introduced in the field of cooperative games as a solution to the problem of distributing the amount $\mu(N)$ among the players. In the area of multiple criteria decision making (MCDM), it reflects the global importance of each criterion.

Definition 2. Let $j \in N$ and let μ be a normalized capacity on N. The Shapley value of criterion j with respect to μ is defined by

$$\phi_j(\mu) = \frac{1}{n}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\Big(\mu(T\cup\{j\}) - \mu(T)\Big).$$

The concepts of k-conjunctive and k-disjunctive functions were introduced for determining the conjunctive/disjunctive character of aggregation functions [26] and they have been applied in the framework of SUOWA operators [27]. k-conjunctive functions are bounded from above by the kth order statistic whereas k-disjunctive functions are bounded from below by the $(n-k+1)$th order statistic.

Definition 3. Let $k \in N$ and let μ be a normalized capacity on N.

1. C_μ is k-conjunctive if $C_\mu \leqslant \text{OS}_k$; i.e., $C_\mu(\boldsymbol{x}) \leqslant x_{(k)}$ for any $\boldsymbol{x} \in \mathbb{R}^n$.
2. C_μ is k-disjunctive if $C_\mu \geqslant \text{OS}_{n-k+1}$; i.e., $C_\mu(\boldsymbol{x}) \geqslant x_{(n-k+1)} = x_{[k]}$ for any $\boldsymbol{x} \in \mathbb{R}^n$.

The set of k-conjunctive (k-disjunctive) Choquet integrals will be denoted by $\mathcal{C}_k$ ($\mathcal{D}_k$). k-conjunctive and k-disjunctive Choquet integrals can be characterized through the values taken by the capacity on subsets of a given cardinality [26].

[4]For the sake of simplicity, we use the notation orness(μ) instead of orness(C_μ). The same comment can be applied to other indices defined later.

Proposition 1. *Let $k \in N$ and let μ be a normalized capacity on N.*

1. $C_\mu \in \mathcal{C}_k$ *if and only if* $\mu(T) = 0$ *for all* $T \subseteq N$ *such that* $|T| \leqslant n - k$.
2. $C_\mu \in \mathcal{D}_k$ *if and only if* $\mu(T) = 1$ *for all* $T \subseteq N$ *such that* $|T| \geqslant k$.

In the case of OWA operators we directly have the following result (see also [22, p. 30]).

Remark 1. Let $k \in N$ and $\boldsymbol{w} \in \mathcal{W}$.

1. $O_{\boldsymbol{w}} \in \mathcal{C}_k$ if and only if $w_i = 0$ for $i = 1, \dots, n - k$.
2. $O_{\boldsymbol{w}} \in \mathcal{D}_k$ if and only if $w_i = 0$ for $i = k + 1, \dots, n$.

Given that k-conjunctive and k-disjunctive Choquet integrals are infrequent in practice, Marichal proposed two indices for measuring the k-conjunctive and k-disjunctive character of Choquet integrals [26].

Definition 4. Given $k \in N \backslash \{n\}$ and μ a normalized capacity, the k-conjunctiveness and k-disjunctiveness indices of C_μ are defined by

$$\mathrm{conj}_k(\mu) = 1 - \frac{1}{n-k} \sum_{t=1}^{n-k} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T| = t}} \mu(T),$$

$$\mathrm{disj}_k(\mu) = \frac{1}{n-k} \sum_{t=k}^{n-1} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T| = t}} \mu(T).$$

Notice that $\mathrm{orness}(\mu) = \mathrm{disj}_1(\mu)$.

The concepts of veto and favor were proposed by Dubois and Koning in the area of the social choice functions [28].

Definition 5. Let $j \in N$ and let μ be a normalized capacity on N.

1. j is a veto for C_μ if $C_\mu(\boldsymbol{x}) \leqslant x_j$ for any $\boldsymbol{x} \in \mathbb{R}^n$.
2. j is a favor for C_μ if $C_\mu(\boldsymbol{x}) \geqslant x_j$ for any $\boldsymbol{x} \in \mathbb{R}^n$.

As in the case of k-conjunctive and k-disjunctive Choquet integrals, veto and favor criteria are infrequent in practice. For this reason Marichal introduced two indices to measure the degree with which a criterion behaves like a veto or a favor [29].

Definition 6. Let $j \in N$ and let μ be a normalized capacity on N. The veto and favor indices of criterion j with respect to μ are defined by

$$\mathrm{veto}_j(\mu) = 1 - \frac{1}{n-1} \sum_{t=1}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \backslash \{j\} \\ |T| = t}} \mu(T),$$

$$\mathrm{favor}_j(\mu) = \frac{1}{n-1} \sum_{t=0}^{n-2} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \backslash \{j\} \\ |T| = t}} \mu(T \cup \{j\}).$$

The following relationship can be established among veto, favor and Shapley value of a criterion [29].

Remark 2. Let $j \in N$ and let μ be a normalized capacity on N. Then,

$$\text{veto}_j(\mu) + \text{favor}_j(\mu) = 1 + \frac{n\phi_j(\mu) - 1}{n - 1}.$$

We finish this subsection showing that, once the indices defined above are known for a normalized capacity, it is immediate to obtain those of its dual capacity (see, for instance, [22, 29]).

Remark 3. If $j \in N$, $k \in N \setminus \{n\}$ and μ is a normalized capacity on N, then

$$\text{orness}(\overline{\mu}) = 1 - \text{orness}(\mu), \qquad \phi_j(\overline{\mu}) = \phi_j(\mu),$$

$$\text{conj}_k(\overline{\mu}) = \text{disj}_k(\mu), \qquad \text{veto}_j(\overline{\mu}) = \text{favor}_j(\mu),$$

$$\text{disj}_k(\overline{\mu}) = \text{conj}_k(\mu), \qquad \text{favor}_j(\overline{\mu}) = \text{veto}_j(\mu).$$

3. The Crescent method

The Crescent method [8] has recently been introduced in the literature to obtain new games, using for this purpose additive and symmetric capacities. In its original formulation, the game is obtained as a convex combination, through the weighting vector $\boldsymbol{p}$, of n previously constructed games. Later, the game obtained with the Crescent method has been expressed as a two-piecewise function where the first piece coincides with a game associated with a SUOWA operator and the second piece is the dual of a game also obtained in the context of SUOWA operators [30].

Definition 7. Let $\boldsymbol{w}, \boldsymbol{p} \in \mathcal{W}$. The game obtained with the Crescent method, $\xi_{\boldsymbol{p},\boldsymbol{w}}$, is given by $\xi_{\boldsymbol{p},\boldsymbol{w}}(\varnothing) = 0$, $\xi_{\boldsymbol{p},\boldsymbol{w}}(N) = 1$, and when $\varnothing \subsetneq A \subsetneq N$,

$$\xi_{\boldsymbol{p},\boldsymbol{w}}(A) = \begin{cases} \dfrac{n}{|A|}\mu_{\boldsymbol{p}}(A)\mu_{|\boldsymbol{w}|}(A) & \text{if } \mu_{|\boldsymbol{w}|}(A) \leqslant \dfrac{|A|}{n}, \\[2ex] \overline{\xi}_{\boldsymbol{p},\overline{\boldsymbol{w}}}(A) = 1 - \xi_{\boldsymbol{p},\overline{\boldsymbol{w}}}(A^{c}) & \text{if } \mu_{|\boldsymbol{w}|}(A) > \dfrac{|A|}{n}, \end{cases}$$

$$= \begin{cases} \dfrac{n}{|A|} \displaystyle\sum_{j \in A} p_j \sum_{k=1}^{|A|} w_k & \text{if } \displaystyle\sum_{k=1}^{|A|} w_k \leqslant \dfrac{|A|}{n}, \\[2ex] 1 - \dfrac{n}{n - |A|} \displaystyle\sum_{j \notin A} p_j \sum_{k=|A|+1}^{n} w_k & \text{if } \displaystyle\sum_{k=1}^{|A|} w_k > \dfrac{|A|}{n}. \end{cases}$$

Remark 4. It is worth noting that, when $\varnothing \subsetneq A \subsetneq N$, the game $\xi_{\boldsymbol{p},\boldsymbol{w}}$ can be also expressed as (see [30])

$$\xi_{\boldsymbol{p},\boldsymbol{w}}(A) = \begin{cases} \dfrac{n}{|A|} \displaystyle\sum_{j \in A} p_j \sum_{k=1}^{|A|} w_k & \text{if } \displaystyle\sum_{k=1}^{|A|} w_k < \dfrac{|A|}{n}, \\[2ex] 1 - \dfrac{n}{n - |A|} \displaystyle\sum_{j \notin A} p_j \sum_{k=|A|+1}^{n} w_k & \text{if } \displaystyle\sum_{k=1}^{|A|} w_k \geqslant \dfrac{|A|}{n}. \end{cases}$$

An outstanding result on these games is that $\xi_{p,\boldsymbol{w}}$ is a capacity when the weighting vector $\boldsymbol{w}$ is unimodal [9]. This result can be extended to a wider family of vectors. We are now going to establish the framework to introduce that family of vectors.

Definition 8. Given $\boldsymbol{w} \in \mathcal{W}$, we define the following elements:

$$L_{\boldsymbol{w}} = \left\{ k \in N \mid \frac{1}{k}\sum_{i=1}^{k} w_i < \frac{1}{n} \right\}, \qquad L^{\boldsymbol{w}} = \left\{ k \in N \mid \frac{1}{k}\sum_{i=1}^{k} w_i > \frac{1}{n} \right\},$$

$$l_{\boldsymbol{w}} = \begin{cases} 0, & \text{if } L_{\boldsymbol{w}} = \varnothing, \\ \max L_{\boldsymbol{w}}, & \text{otherwise,} \end{cases} \qquad l^{\boldsymbol{w}} = \begin{cases} n, & \text{if } L^{\boldsymbol{w}} = \varnothing, \\ \min L^{\boldsymbol{w}}, & \text{otherwise.} \end{cases}$$

Although the elements $L_{\boldsymbol{w}}$, $l_{\boldsymbol{w}}$, $L^{\boldsymbol{w}}$, and $l^{\boldsymbol{w}}$ were originally defined only for unimodal weighting vectors [23], they can be defined for any weighting vector. Moreover, we have modified the definition of $l^{\boldsymbol{w}}$ when $L^{\boldsymbol{w}} = \varnothing$ (changing the value from $n + 1$ to n) to make some proofs easier. For instance, the following lemma, that shows some relationships between the indices $l_{\boldsymbol{w}}$ and $l^{\boldsymbol{w}}$ of a weighting vector and its dual, is given in a more compact form than in its original formulation [9].

Lemma 1. *Let $\boldsymbol{w} \in \mathcal{W}$. Then*

1. $l_{\boldsymbol{w}} + l^{\overline{\boldsymbol{w}}} = n.$
2. $l^{\boldsymbol{w}} + l_{\overline{\boldsymbol{w}}} = n.$

Proof. See the proof of Lemma 2 in [9], taking into account that now $l^{\boldsymbol{w}} = n$ when $L^{\boldsymbol{w}} = \varnothing$. □

As we have previously commented, the game $\xi_{p,\boldsymbol{w}}$ is a capacity when the weighting vector $\boldsymbol{w}$ is unimodal. The proof of this result is based on the following properties of unimodal weighting vectors [9]:

(P1) $l_{\boldsymbol{w}} < l^{\boldsymbol{w}}$.

(P2) If $L_{\boldsymbol{w}} \neq \varnothing$, then $L_{\boldsymbol{w}} = \{1, \dots, l_{\boldsymbol{w}}\}$.

(P3) If $L^{\boldsymbol{w}} \neq \varnothing$, then $L^{\boldsymbol{w}} = \{l^{\boldsymbol{w}}, \dots, n-1\}$.

(P4) If $p, q \in N$ with $p < q \leqslant l_{\boldsymbol{w}}$, then

$$\frac{1}{p}\sum_{i=1}^{p} w_i \leqslant \frac{1}{q}\sum_{i=1}^{q} w_i.$$

(P5) If $p, q \in N$ with $p < q \leqslant l_{\overline{\boldsymbol{w}}}$, then

$$\frac{1}{p}\sum_{i=1}^{p} \overline{w}_i \leqslant \frac{1}{q}\sum_{i=1}^{q} \overline{w}_i.$$

Note that (P1) is a consequence of (P2) and that (P4) implies (P2). Furthermore (P5) also implies (P3), since if $p > l^{\boldsymbol{w}}$, $p \neq n$, then $n - p < n - l^{\boldsymbol{w}} = l_{\overline{\boldsymbol{w}}}$. Therefore

$$\frac{1}{n-p}\sum_{i=1}^{n-p} \overline{w}_i < \frac{1}{n}.$$

But,

$$\frac{1}{n-p}\sum_{i=1}^{n-p}\overline{w}_i < \frac{1}{n} \;\Leftrightarrow\; \sum_{i=n-p+1}^{n}\overline{w}_i > \frac{p}{n} \;\Leftrightarrow\; \frac{1}{p}\sum_{i=1}^{p} w_i > \frac{1}{n} \;\Leftrightarrow\; p \in L^{\boldsymbol{w}}.$$

Hence, properties (P1)–(P5) can be guaranteed when considering the following family of weighting vectors.

Definition 9. $\mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$ is the set of weighting vectors $\boldsymbol{w}$ that satisfy the following conditions:

(C1) If $p, q \in N$ with $p < q \leqslant l_{\boldsymbol{w}}$, then

$$\frac{1}{p}\sum_{i=1}^{p} w_i \leqslant \frac{1}{q}\sum_{i=1}^{q} w_i.$$

(C2) If $p, q \in N$ with $p < q \leqslant l_{\overline{\boldsymbol{w}}}$, then

$$\frac{1}{p}\sum_{i=1}^{p} \overline{w}_i \leqslant \frac{1}{q}\sum_{i=1}^{q} \overline{w}_i.$$

Notice that (C2) can be written alternatively in terms of $\boldsymbol{w}$: If $p, q \in N$ with $p < q \leqslant n - l^{\boldsymbol{w}}$, then

$$\frac{1}{p}\sum_{i=1}^{p} w_{n+1-i} \leqslant \frac{1}{q}\sum_{i=1}^{q} w_{n+1-i}.$$

Moreover, conditions (C1) and (C2) can be written in a more compact way if we use the vectors of means of $\boldsymbol{w}$ and $\overline{\boldsymbol{w}}$; where, given a vector $\boldsymbol{w}$, the vector of means $\overset{\triangleleft}{\boldsymbol{w}}$ is defined by

$$\overset{\triangleleft}{w}_j = \frac{1}{j}\sum_{i=1}^{j} w_i, \qquad j = 1, \dots, n.$$

Hence, conditions (C1) and (C2) become, respectively, $\overset{\triangleleft}{w}_p \leqslant \overset{\triangleleft}{w}_q$ when $p < q \leqslant l_{\boldsymbol{w}}$ and $\overset{\triangleleft}{\overline{w}}_p \leqslant \overset{\triangleleft}{\overline{w}}_q$ when $p < q \leqslant l_{\overline{\boldsymbol{w}}}$. Note also that the vector of means makes it easy to check whether a vector belongs to the set $\mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$. For instance,

1. Consider $\boldsymbol{w} = (0.4, 0.2, 0.3, 0.1)$. Then $\overset{\triangleleft}{\boldsymbol{w}} = (0.4, 0.3, 0.3, 0.25)$, and $l_{\boldsymbol{w}} = 0$. On the other hand, $\overline{\boldsymbol{w}} = (0.1, 0.3, 0.2, 0.4)$, $\overset{\triangleleft}{\overline{\boldsymbol{w}}} = (0.1, 0.2, 0.2, 0.25)$, and $l_{\overline{\boldsymbol{w}}} = 3$. Since $\overset{\triangleleft}{\overline{w}}_1 \leqslant \overset{\triangleleft}{\overline{w}}_2 \leqslant \overset{\triangleleft}{\overline{w}}_3$, we have that $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$.
2. Consider $\boldsymbol{w} = (0.4, 0.1, 0.4, 0.1)$. Then $\overset{\triangleleft}{\boldsymbol{w}} = (0.4, 0.25, 0.3, 0.25)$, and $l_{\boldsymbol{w}} = 0$. On the other hand, $\overline{\boldsymbol{w}} = (0.1, 0.4, 0.1, 0.4)$, $\overset{\triangleleft}{\overline{\boldsymbol{w}}} = (0.1, 0.25, 0.2, 0.25)$, and $l_{\overline{\boldsymbol{w}}} = 3$. Since $\overset{\triangleleft}{\overline{w}}_2 > \overset{\triangleleft}{\overline{w}}_3$, we have that $\boldsymbol{w} \notin \mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$.

It is also worth noting that $\mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$ is the set of weighting vectors $\boldsymbol{w}$ such that both $\boldsymbol{w}$ and $\overline{\boldsymbol{w}}$ belong to the set

$$\left\{\boldsymbol{w} \in \mathcal{W} \mid \overset{\triangleleft}{w}_p \leqslant \overset{\triangleleft}{w}_q \text{ for any } p < q \leqslant l_{\boldsymbol{w}}\right\}.$$

From this it follows that $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$ if and only if $\overline{\boldsymbol{w}} \in \mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$.

In addition to the unimodal weighting vectors, an outstanding family of vectors that belong to the set $\mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$ are those that give rise to the Winsorized means (see Section 5).

Lastly, taking into account the previous comments we have the following results.

Proposition 2. *If $\boldsymbol{p} \in \mathcal{W}$ and $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{L}}$, then $\xi_{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity on N given by $\xi_{\boldsymbol{p},\boldsymbol{w}}(\varnothing) = 0$, $\xi_{\boldsymbol{p},\boldsymbol{w}}(N) = 1$, and when $\varnothing \subsetneq A \subsetneq N$,*

$$\xi_{\boldsymbol{p},\boldsymbol{w}}(A) = \begin{cases} \dfrac{n}{|A|} \displaystyle\sum_{j \in A} p_j \sum_{k=1}^{|A|} w_k & \text{if } |A| < l^{\boldsymbol{w}}, \\ 1 - \dfrac{n}{n - |A|} \displaystyle\sum_{j \notin A} p_j \sum_{k=|A|+1}^{n} w_k & \text{if } |A| \geqslant l^{\boldsymbol{w}}, \end{cases}$$

$$= \begin{cases} n\, \overset{\lhd}{w}_{|A|} \displaystyle\sum_{j \in A} p_j & \text{if } |A| < l^{\boldsymbol{w}}, \\ 1 - n\, \overset{\lhd}{\overline{w}}_{n-|A|} \displaystyle\sum_{j \notin A} p_j & \text{if } |A| \geqslant l^{\boldsymbol{w}}. \end{cases}$$

Remark 5. According to Remark 4, when $\varnothing \subsetneq A \subsetneq N$, the value $\xi_{\boldsymbol{p},\boldsymbol{w}}(A)$ can be also expressed in terms of $l_{\boldsymbol{w}}$ as follows:

$$\xi_{\boldsymbol{p},\boldsymbol{w}}(A) = \begin{cases} n\, \overset{\lhd}{w}_{|A|} \displaystyle\sum_{j \in A} p_j & \text{if } |A| \leqslant l_{\boldsymbol{w}}, \\ 1 - n\, \overset{\lhd}{\overline{w}}_{n-|A|} \displaystyle\sum_{j \notin A} p_j & \text{if } |A| > l_{\boldsymbol{w}}. \end{cases}$$

The Choquet integral with respect to $\xi_{\boldsymbol{p},\boldsymbol{w}}$ will be denoted by $C_{\boldsymbol{p},\boldsymbol{w}}$.

4. The results

In this section we give the main results of the paper. In the first subsection we analyze the conjunctive/disjunctive character of $C_{\boldsymbol{p},\boldsymbol{w}}$ while in the second subsection we give closed-form expressions for the veto and favor indices, and the Shapley values.

The following remark will be useful in the proofs of some of the results established in the subsequent subsections.

Remark 6. Let $\boldsymbol{p}$ be a weighting vector. If $t \geqslant 1$, then

$$\sum_{\substack{T \subseteq N \\ |T|=t}} \sum_{i \in T} p_i = \binom{n-1}{t-1} \sum_{i=1}^{n} p_i = \binom{n-1}{t-1} = \binom{n}{t} \frac{t}{n},$$

$$\sum_{\substack{T \subseteq N \\ |T|=t}} \sum_{i \notin T} p_i = \sum_{\substack{T \subseteq N \\ |T|=t}} \left(1 - \sum_{i \in T} p_i\right) = \binom{n}{t} \frac{n-t}{n},$$

and, for any $j \in N$,

$$\sum_{\substack{T \subseteq N \setminus \{j\} \\ |T|=t}} \sum_{i \in T} p_i = \binom{n-2}{t-1} \sum_{\substack{i=1 \\ i \neq j}}^{n} p_i = \binom{n-2}{t-1} (1 - p_j) = \binom{n-1}{t} \frac{t(1-p_j)}{n-1},$$

$$\sum_{\substack{T \subseteq N \setminus \{j\} \\ |T|=t}} \sum_{i \notin T} p_i = \sum_{\substack{T \subseteq N \setminus \{j\} \\ |T|=t}} \left(1 - \sum_{i \in T} p_i\right) = \binom{n-1}{t} \left(1 - \frac{t(1-p_j)}{n-1}\right).$$

4.1. Conjunctive/disjunctive character of $C_{\boldsymbol{p},\boldsymbol{w}}$

The first result of this subsection shows that the k-conjunctive or k-disjunctive character of OWA operators is preserved by $C_{\boldsymbol{p},\boldsymbol{w}}$.

Proposition 3. *Let $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{L}}$.*

1. *If $k \in N$ and $O_{\boldsymbol{w}} \in \mathcal{C}_k$, then $C_{\boldsymbol{p},\boldsymbol{w}} \in \mathcal{C}_k$ for any weighting vector $\boldsymbol{p}$.*
2. *If $k \in N$ and $O_{\boldsymbol{w}} \in \mathcal{D}_k$, then $C_{\boldsymbol{p},\boldsymbol{w}} \in \mathcal{D}_k$ for any weighting vector $\boldsymbol{p}$.*

Proof. The proofs are immediate taking into account Remark 1 and Propositions 1 and 2. □

From the previous results it is immediately clear that the operator $C_{\boldsymbol{p},\boldsymbol{w}}$ is located between two order statistics whenever the corresponding OWA operator also ranges between the same order statistics.

Corollary 1. *Let $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{L}}$ such that there exist $k, k' \in N$ with $\mathrm{OS}_k \leqslant O_{\boldsymbol{w}} \leqslant \mathrm{OS}_{k'}$. Then $\mathrm{OS}_k \leqslant C_{\boldsymbol{p},\boldsymbol{w}} \leqslant \mathrm{OS}_{k'}$ for any weighting vector $\boldsymbol{p}$.*

Regarding the k-conjunctiveness and k-disjunctiveness indices of $C_{\boldsymbol{p},\boldsymbol{w}}$, the values of these indices coincide with the respective indices of the OWA operator.

Proposition 4. *Let $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{L}}$ and $\boldsymbol{p} \in \mathcal{W}$. Then, for any $k \in N\backslash\{n\}$, we have*

$$\mathrm{conj}_k\left(\xi_{\boldsymbol{p},\boldsymbol{w}}\right) = \mathrm{conj}_k(\mu_{|\boldsymbol{w}|}), \quad \mathrm{disj}_k\left(\xi_{\boldsymbol{p},\boldsymbol{w}}\right) = \mathrm{disj}_k(\mu_{|\boldsymbol{w}|}).$$

Proof. Given $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{L}}$, $\boldsymbol{p} \in \mathcal{W}$ and $k \in N\backslash\{n\}$, let $q = \min(n-k, l^{\boldsymbol{w}} - 1)$. By Proposition 2 and Remark 6 we have:

$$\begin{aligned}
&\frac{1}{n-k}\sum_{t=1}^{n-k}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\xi_{\boldsymbol{p},\boldsymbol{w}}(T) = \\
&\quad= \frac{1}{n-k}\left(\sum_{t=1}^{q}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\frac{n}{t}\left(\sum_{i=1}^{t}w_i\right)\sum_{i\in T}p_i + \sum_{t=l^{\boldsymbol{w}}}^{n-k}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\left(1-\frac{n}{n-t}\left(\sum_{i=t+1}^{n}w_i\right)\sum_{i\notin T}p_i\right)\right)\\
&\quad= \frac{1}{n-k}\left(\sum_{t=1}^{q}\frac{1}{\binom{n}{t}}\frac{n}{t}\left(\sum_{i=1}^{t}w_i\right)\binom{n}{t}\frac{t}{n} + \sum_{t=l^{\boldsymbol{w}}}^{n-k}\frac{1}{\binom{n}{t}}\left(\binom{n}{t} - \frac{n}{n-t}\left(\sum_{i=t+1}^{n}w_i\right)\binom{n}{t}\frac{n-t}{n}\right)\right)\\
&\quad= \frac{1}{n-k}\left(\sum_{t=1}^{q}\sum_{i=1}^{t}w_i + \sum_{t=l^{\boldsymbol{w}}}^{n-k}\left(1-\sum_{i=t+1}^{n}w_i\right)\right) = \frac{1}{n-k}\sum_{t=1}^{n-k}\sum_{i=1}^{t}w_i\\
&\quad= \frac{1}{n-k}\sum_{i=1}^{n-k}(n+1-k-i)w_i.
\end{aligned}$$

Therefore

$$\mathrm{conj}_k\left(\xi_{\boldsymbol{p},\boldsymbol{w}}\right) = 1 - \frac{1}{n-k}\sum_{t=1}^{n-k}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\xi_{\boldsymbol{p},\boldsymbol{w}}(T) = 1 - \frac{1}{n-k}\sum_{i=1}^{n-k}(n+1-k-i)w_i = \mathrm{conj}_k(\mu_{|\boldsymbol{w}|}).$$

On the other hand,

$$\mathrm{disj}_k\left(\xi_{\boldsymbol{p},\boldsymbol{w}}\right) = \mathrm{conj}_k\left(\overline{\xi}_{\boldsymbol{p},\boldsymbol{w}}\right) = \mathrm{conj}_k\left(\xi_{\boldsymbol{p},\overline{\boldsymbol{w}}}\right) = \mathrm{conj}_k(\mu_{|\overline{\boldsymbol{w}}|}) = \mathrm{disj}_k(\mu_{|\boldsymbol{w}|}).$$ □

4.2. Veto, favor and Shapley values

In this subsection we show closed-form expressions for the veto and favor indices,[5] and the Shapley values. It is worth noting that, although the expressions obtained for the veto and favor indices are complex, the expression for the Shapley value is relatively simple.

Proposition 5. *Let $\mathbf{w} \in \mathcal{W}_{\overleftarrow{\mathrm{L}}}$ and $\mathbf{p} \in \mathcal{W}$. Then, for any $j \in N$, we have*

$$\mathrm{veto}_j(\xi_{\mathbf{p},\mathbf{w}}) = 1-\frac{1}{n-1}\left(n-l^{\mathbf{w}}+\frac{n}{n-1}\left((1-p_j)\left(\sum_{k=1}^{l^{\mathbf{w}}-1}\sum_{i=1}^{k}w_i-\sum_{k=1}^{n-l^{\mathbf{w}}}\sum_{i=1}^{k}w_{n+1-i}\right)-(np_j-1)\sum_{k=1}^{n-l^{\mathbf{w}}}\frac{1}{k}\sum_{i=1}^{k}w_{n+1-i}\right)\right).$$

Proof. By Proposition 2 and Remark 6 we have

$$\sum_{t=1}^{l^{\mathbf{w}}-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\xi_{\mathbf{p},\mathbf{w}}(T) = \sum_{t=1}^{l^{\mathbf{w}}-1}\frac{1}{\binom{n-1}{t}}\frac{n}{t}\left(\sum_{i=1}^{t}w_i\right)\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\sum_{i\in T}p_i = \frac{n}{n-1}(1-p_j)\sum_{t=1}^{l^{\mathbf{w}}-1}\sum_{i=1}^{t}w_i.$$

Analogously,

$$\begin{aligned}
\sum_{t=l^{\mathbf{w}}}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\xi_{\mathbf{p},\mathbf{w}}(T) &= \sum_{t=l^{\mathbf{w}}}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\left(1-\frac{n}{n-t}\left(\sum_{i=t+1}^{n}w_i\right)\sum_{i\notin T}p_i\right)\\
&= \sum_{t=l^{\mathbf{w}}}^{n-1}\left(1-\frac{n}{n-t}\left(\sum_{i=t+1}^{n}w_i\right)\left(1-\frac{t(1-p_j)}{n-1}\right)\right)\\
&= n-l^{\mathbf{w}}-\frac{n}{n-1}\sum_{t=l^{\mathbf{w}}}^{n-1}\left(\sum_{i=t+1}^{n}w_i\right)\frac{n-1-t+tp_j}{n-t}\\
&= n-l^{\mathbf{w}}-\frac{n}{n-1}\sum_{k=1}^{n-l^{\mathbf{w}}}\left(\sum_{i=n-k+1}^{n}w_i\right)\left(1-p_j+\frac{np_j-1}{k}\right)\\
&= n-l^{\mathbf{w}}-\frac{n}{n-1}\left((1-p_j)\sum_{k=1}^{n-l^{\mathbf{w}}}\sum_{i=1}^{k}w_{n+1-i}+(np_j-1)\sum_{k=1}^{n-l^{\mathbf{w}}}\frac{1}{k}\sum_{i=1}^{k}w_{n+1-i}\right)
\end{aligned}$$

Now, taking into account that

$$\mathrm{veto}_j(\xi_{\mathbf{p},\mathbf{w}}) = 1-\frac{1}{n-1}\left(\sum_{t=1}^{l^{\mathbf{w}}-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\xi_{\mathbf{p},\mathbf{w}}(T)+\sum_{t=l^{\mathbf{w}}}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\xi_{\mathbf{p},\mathbf{w}}(T)\right),$$

we get the result. □

We next give closed-form expressions for the Shapley values by using the relationship between the veto and favor indices and the Shapley value (see Remark 2).

Theorem 1. *Let $\mathbf{w} \in \mathcal{W}_{\overleftarrow{\mathrm{L}}}$ and $\mathbf{p} \in \mathcal{W}$. Then, for any $j \in N$, we have*

$$\phi_j(\xi_{\mathbf{p},\mathbf{w}}) = \frac{1}{n-1}\left(1-p_j+(np_j-1)W\right),$$

[5] The expression for the favor index is obtained from the veto index of the dual capacity (see Remark 3).

where

$$W = \sum_{k=1}^{l_{\boldsymbol{w}}} \overset{\triangleleft}{w}_k + \sum_{k=1}^{n-l^{\boldsymbol{w}}} \overset{\triangleleft}{\overline{w}}_k + \frac{l^{\boldsymbol{w}} - l_{\boldsymbol{w}}}{n}. \tag{1}$$

Proof. Since $\mathrm{favor}_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) = \mathrm{veto}_j(\overline{\xi}_{\boldsymbol{p},\boldsymbol{w}}) = \mathrm{veto}_j(\xi_{\boldsymbol{p},\overline{\boldsymbol{w}}})$ and $l^{\overline{\boldsymbol{w}}} = n - l_{\boldsymbol{w}}$, by Proposition 5 we have

$$\mathrm{favor}_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) = 1 - \frac{1}{n-1}\left(l_{\boldsymbol{w}} + \frac{n}{n-1}\left((1-p_j)\left(\sum_{k=1}^{n-1-l_{\boldsymbol{w}}} \sum_{i=1}^{k} w_{n+1-i} - \sum_{k=1}^{l_{\boldsymbol{w}}} \sum_{i=1}^{k} w_i\right) - (np_j - 1)\sum_{k=1}^{l_{\boldsymbol{w}}} \frac{1}{k} \sum_{i=1}^{k} w_i\right)\right).$$

Now, by using the notation

$$W' = \sum_{k=1}^{l_{\boldsymbol{w}}} \overset{\triangleleft}{w}_k + \sum_{k=1}^{n-l^{\boldsymbol{w}}} \overset{\triangleleft}{\overline{w}}_k = \sum_{k=1}^{l_{\boldsymbol{w}}} \frac{1}{k} \sum_{i=1}^{k} w_i + \sum_{k=1}^{n-l^{\boldsymbol{w}}} \frac{1}{k} \sum_{i=1}^{k} w_{n+1-i},$$

and taking into account that

$$\begin{aligned}
&\sum_{k=1}^{l^{\boldsymbol{w}}-1} \sum_{i=1}^{k} w_i - \sum_{k=1}^{n-l^{\boldsymbol{w}}} \sum_{i=1}^{k} w_{n+1-i} + \sum_{k=1}^{n-1-l_{\boldsymbol{w}}} \sum_{i=1}^{k} w_{n+1-i} - \sum_{k=1}^{l_{\boldsymbol{w}}} \sum_{i=1}^{k} w_i = \sum_{k=l_{\boldsymbol{w}}+1}^{l^{\boldsymbol{w}}-1} \sum_{i=1}^{k} w_i + \sum_{k=n-l^{\boldsymbol{w}}+1}^{n-1-l_{\boldsymbol{w}}} \sum_{i=1}^{k} w_{n+1-i} \\
&= \sum_{t=l_{\boldsymbol{w}}+1}^{l^{\boldsymbol{w}}-1} \sum_{i=1}^{t} w_i + \sum_{t=l_{\boldsymbol{w}}+1}^{l^{\boldsymbol{w}}-1} \sum_{i=1}^{n-t} w_{n+1-i} = \sum_{t=l_{\boldsymbol{w}}+1}^{l^{\boldsymbol{w}}-1} \left(\sum_{i=1}^{t} w_i + \sum_{i=t+1}^{n} w_i\right) = l^{\boldsymbol{w}} - l_{\boldsymbol{w}} - 1,
\end{aligned}$$

we have

$$\begin{aligned}
\phi_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) &= \frac{1 + (n-1)(\mathrm{veto}_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) + \mathrm{favor}_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) - 1)}{n} \\
&= \frac{l^{\boldsymbol{w}} - l_{\boldsymbol{w}}}{n} - \frac{1}{n-1}\Big((1-p_j)(l^{\boldsymbol{w}} - l_{\boldsymbol{w}} - 1) - (np_j - 1)W'\Big) \\
&= \frac{1}{n-1}\left(\frac{n-1}{n}(l^{\boldsymbol{w}} - l_{\boldsymbol{w}}) - (1-p_j)(l^{\boldsymbol{w}} - l_{\boldsymbol{w}}) + 1 - p_j + (np_j - 1)W'\right) \\
&= \frac{1}{n-1}\left(\left(p_j - \frac{1}{n}\right)(l^{\boldsymbol{w}} - l_{\boldsymbol{w}}) + 1 - p_j + (np_j - 1)W'\right) \\
&= \frac{1}{n-1}\Big(1 - p_j + (np_j - 1)W\Big).
\end{aligned}$$

□

In the following proposition we show some properties of the value W, which will be useful in the proof of some later results.

Proposition 6. *Let $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$ and*

$$W = \sum_{k=1}^{l_{\boldsymbol{w}}} \overset{\triangleleft}{w}_k + \sum_{k=1}^{n-l^{\boldsymbol{w}}} \overset{\triangleleft}{\overline{w}}_k + \frac{l^{\boldsymbol{w}} - l_{\boldsymbol{w}}}{n}.$$

Then:

1. *$W \geqslant 1/n$, and $W = 1/n$ if and only if $\boldsymbol{w} = \boldsymbol{e}_k$ for some $k \in N$.*
2. *$W \leqslant 1$, and $W = 1$ if and only if $\boldsymbol{w} = \boldsymbol{\eta}$.*

Proof.

1. Since $l^{w} > l_{w}$ we have $W \geqslant 1/n$. Moreover, $W = 1/n$ if and only if $l^{w} = l_{w} + 1$, $w_i = 0$ for any $i \in \{1, \ldots, l_{w}\}$, and $w_{n+1-i} = 0$ for any $i \in \{1, \ldots, n - l^{w}\}$ (that is, $w_i = 0$ for any $i \in \{l^{w} + 1, \ldots, n\}$). Therefore, $W = 1/n$ if and only if there exists $k \in N$ such that $\boldsymbol{w} = \boldsymbol{e}_k$.
2. Notice that if $k \in \{1, \ldots, n - l^{w}\}$, then $n - k \in \{l^{w}, \ldots, n-1\} = L^{w}$. From the definition of L^{w} we have that

$$\frac{1}{n-k}\sum_{i=1}^{n-k} w_i > \frac{1}{n} \;\Leftrightarrow\; 1 - \sum_{i=n-k+1}^{n} w_i > \frac{n-k}{n} \;\Leftrightarrow\; \sum_{i=n-k+1}^{n} w_i < \frac{k}{n} \;\Leftrightarrow\; \frac{1}{k}\sum_{i=1}^{k} w_{n+1-i} < \frac{1}{n}.$$

Therefore,

$$W \leqslant l_{w}\frac{1}{n} + (n - l^{w})\frac{1}{n} + \frac{l^{w} - l_{w}}{n} = 1.$$

Notice also that if $l_{w} > 0$ or $l^{w} < n$ then $W < 1$, or equivalently, if $W = 1$ then $l_{w} = 0$ and $l^{w} = n$; that is, $\boldsymbol{w} = \boldsymbol{\eta}$. □

Notice that the Shapley value can be expressed as

$$\begin{aligned}\phi_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) &= \frac{1}{n-1}\Big(1 - p_j + (np_j - 1)W\Big)\\ &= \frac{1}{n-1}\Big(p_j(nW-1) + 1 - W\Big) \qquad (2)\\ &= \frac{nW-1}{n-1}p_j + \Big(1 - \frac{nW-1}{n-1}\Big)\frac{1}{n}, \qquad (3)\end{aligned}$$

and given that, from Proposition 6 we get that $0 \leqslant (nW-1)/(n-1) \leqslant 1$, we have that the Shapley value is a convex combination between p_j and $1/n$, which are the Shapley values of the capacities of the weighted mean $M_{\boldsymbol{p}}$ and any OWA operator, respectively. This means that the Shapley value is less than p_j when $p_j > 1/n$, and greater than p_j when $p_j < 1/n$. Note also that $\phi_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) = p_j$ for any $j \in N$ and any weighting vector $\boldsymbol{p}$ if and only if $W = 1$; that is, when $\boldsymbol{w} = \boldsymbol{\eta}$.

On the other hand, since the Shapley values allow us to know the global importance of each criterion, it is essential to be able to determine the weights that allow us to obtain Shapley values previously fixed. Hence, when $W \neq 1/n$, from expression (2) we can give the weight p_j in terms of $\phi_j(\xi_{\boldsymbol{p},\boldsymbol{w}})$:

$$p_j = \frac{(n-1)\phi_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) + W - 1}{nW - 1}.$$

From the above expression it is easy to check that $p_j \geqslant 0$ if and only if $\phi_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) \geqslant \frac{1-W}{n-1}$, and that $\sum_{j=1}^{n} p_j = 1$. Thus, we have the following corollary.

Corollary 2. *Let $(\phi_1, \ldots, \phi_n)$ be a weighting vector and let $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$ such that $\boldsymbol{w} \neq \boldsymbol{e}_k$ for any $k \in N$. Then the following conditions are equivalent:*

1. $\min_{j \in N} \phi_j \geqslant \dfrac{1-W}{n-1}$.
2. *The vector $\boldsymbol{p}$ defined by*

$$p_j = \frac{(n-1)\phi_j + W - 1}{nW - 1}, \quad j = 1, \ldots, n,$$

is a weighting vector such that $\phi_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) = \phi_j$ for any $j \in N$.

5. Particular cases

The results given in the previous section are valid for any weighting vector $\boldsymbol{w}$ that belongs to $\mathcal{W}_{\overleftrightarrow{L}}$. Within this set there are several very interesting families of vectors that will be studied in more detail below. The first two families encompass nondecreasing and nonincreasing weighting vectors, respectively:

1. If $\boldsymbol{w}$ is a weighting vector different from $\boldsymbol{e}_n$ such that the vector $\overset{\triangleleft}{\boldsymbol{w}}$ is nondecreasing, then $l^{\boldsymbol{w}} = n$, and $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{L}}$. By Proposition 2,

$$\xi_{\boldsymbol{p},\boldsymbol{w}}(A) = n\,\overset{\triangleleft}{w}_{|A|} \sum_{j\in A} p_j$$

for any $A \subseteq N$, $A \neq \varnothing$, and the thesis of Corollary 2 is valid, being

$$W = \sum_{k=1}^{l_{\boldsymbol{w}}} \overset{\triangleleft}{w}_k + \frac{n - l_{\boldsymbol{w}}}{n} = \sum_{k=1}^{n} \overset{\triangleleft}{w}_k,$$

that is, W is the sum of the components of the vector $\overset{\triangleleft}{\boldsymbol{w}}$. Notice that the last equality is fulfilled because, when $k > l_{\boldsymbol{w}}$, we have $\overset{\triangleleft}{w}_k = 1/n$.

2. A similar result can be obtained for any weighting vector $\boldsymbol{w}$ such that its dual satisfies the above conditions. In this case, the result can be written as follows. If $\boldsymbol{w}$ is a weighting vector different from $\boldsymbol{e}_1$ such that $\overset{\triangleleft}{\overline{\boldsymbol{w}}}$ is nondecreasing (or, equivalently, the sequence $\left(\frac{1}{n-k+1}\sum_{i=k}^{n} w_i\right)_{k=1}^{n}$ is nonincreasing), then $l_{\boldsymbol{w}} = 0$ and $\boldsymbol{w} \in \mathcal{W}_{\overleftrightarrow{L}}$. By Remark 5,

$$\xi_{\boldsymbol{p},\boldsymbol{w}}(A) = 1 - n\,\overset{\triangleleft}{\overline{w}}_{n-|A|} \sum_{j\notin A} p_j$$

for any $A \subsetneq N$, and the thesis of Corollary 2 is valid, being

$$W = \sum_{k=1}^{n-l^{\boldsymbol{w}}} \overset{\triangleleft}{\overline{w}}_k + \frac{l^{\boldsymbol{w}}}{n} = \sum_{k=1}^{n} \overset{\triangleleft}{\overline{w}}_k.$$

Now we analyze the families of vectors that give rise to Winsorized ([31, 32]) and trimmed means. For this, we consider the following set:

$$\mathcal{R} = \{(r,s) \in \{0,1,\dots,n-1\}^2 \mid r+s \leqslant n-1\}.$$

The weighting vectors that give rise to the Winsorized means are defined as follows.

Definition 10. Given $(r,s) \in \mathcal{R}$, the weighting vector $\boldsymbol{w}^{(r,s)}$ is defined by

$$w_i^{(r,s)} = \begin{cases} 0 & \text{if } i = 1,\dots,s, \\ \frac{s+1}{n} & \text{if } i = s+1, \\ \frac{1}{n} & \text{if } i = s+2,\dots,n-r-1, \\ \frac{r+1}{n} & \text{if } i = n-r, \\ 0 & \text{otherwise,} \end{cases}$$

when $r + s \leqslant n - 2$, and $\boldsymbol{w}^{(r,s)} = \boldsymbol{e}_{s+1}$ when $r + s = n - 1$.

It is easy to check that $l_{\boldsymbol{w}^{(r,s)}} = s$, $l^{\boldsymbol{w}^{(r,s)}} = n - r$, and $\boldsymbol{w}^{(r,s)} \in \mathcal{W}_{\overleftrightarrow{\mathrm{L}}}$ (notice also that $\boldsymbol{w}^{(r,s)}$ is not unimodal when $r + s \leqslant n - 3$ and $\min(r, s) \geqslant 1$). Therefore, for any weighting vector $\boldsymbol{p}$, $\xi_{\boldsymbol{p},\boldsymbol{w}^{(r,s)}}$ is a normalized capacity on N given by

$$\xi_{\boldsymbol{p},\boldsymbol{w}^{(r,s)}}(A) = \begin{cases} 0, & \text{if } |A| \leqslant s, \\ \sum_{i \in A} p_i, & \text{if } s < |A| < n - r, \\ 1, & \text{if } |A| \geqslant n - r. \end{cases}$$

The Choquet integral with respect to $\xi_{\boldsymbol{p},\boldsymbol{w}^{(r,s)}}$ is the (r, s)-fold Winsorized weighted mean (see [23, 33]), which is defined by

$$M_{\boldsymbol{p}}^{(r,s)}(\boldsymbol{x}) = \left(\sum_{i=1}^{s} p_{[i]}\right) x_{[s+1]} + \sum_{i=s+1}^{n-r} p_{[i]} x_{[i]} + \left(\sum_{i=n-r+1}^{n} p_{[i]}\right) x_{[n-r]}.$$

Notice that in these operators, the s highest values and the r lowest values of a vector of values $\boldsymbol{x}$ are replaced by $x_{[s+1]}$ and $x_{[n-r]}$, respectively, and after that, the weighted mean associated with a weighting vector $\boldsymbol{p}$ is considered. They are the natural generalization of the Winsorized means and were obtained in the framework of the SUOWA operators by using the weighting vector $\boldsymbol{w}^{(r,s)}$ and the semiuninorm $U_{\min}^{\max}$.

Regarding the Shapley value, notice that $W = (n - r - s)/n$ and, therefore, expression (3) becomes

$$\phi_j(\xi_{\boldsymbol{p},\boldsymbol{w}}) = \left(1 - \frac{r + s}{n - 1}\right) p_j + \frac{r + s}{n - 1} \frac{1}{n}.$$

The trimmed means are obtained using the following family of weighting vectors.

Definition 11. Given $(r, s) \in \mathcal{R}$, the weighting vector $\widetilde{\boldsymbol{w}}^{(r,s)}$ is defined by

$$\widetilde{w}_i^{(r,s)} = \begin{cases} 0 & \text{if } i = 1, \ldots, s, \\ \frac{1}{n-(r+s)} & \text{if } i = s + 1, \ldots, n - r, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the weighting vectors $\widetilde{\boldsymbol{w}}^{(r,s)}$ are unimodal, and that, when $\min(r, s) \geqslant 1$, we have $l_{\widetilde{\boldsymbol{w}}^{(r,s)}} = \left\lceil \frac{ns}{r+s} \right\rceil - 1$, and $l^{\widetilde{\boldsymbol{w}}^{(r,s)}} = \left\lfloor \frac{ns}{r+s} \right\rfloor + 1$ (see [23]).[6] Therefore, according to Proposition 2, for any weighting vector $\boldsymbol{p}$ and $(r, s) \in \mathcal{R}$

[6]Note that when $\min(r, s) < 1$, the vector $\widetilde{\boldsymbol{w}}^{(r,s)}$ belongs to one of the families seen above. Specifically, when $s = 0$ and $r \geqslant 1$, the vector $\widetilde{\boldsymbol{w}}^{(r,s)}$ is nonincreasing with $\widetilde{w}_1^{(r,s)} > 1/n$. Therefore $l_{\widetilde{\boldsymbol{w}}^{(r,0)}} = 0$, and $l^{\widetilde{\boldsymbol{w}}^{(r,0)}} = 1$. When $s \geqslant 1$ and $r = 0$, the vector $\widetilde{\boldsymbol{w}}^{(r,s)}$ is nondecreasing with $\widetilde{w}_1^{(r,s)} < 1/n$. Therefore $l_{\widetilde{\boldsymbol{w}}^{(r,0)}} = n - 1$, and $l^{\widetilde{\boldsymbol{w}}^{(r,0)}} = n$. When $s = 0$ and $r = 0$, $\widetilde{\boldsymbol{w}}^{(0,0)} = \boldsymbol{\eta}$; hence $l_{\widetilde{\boldsymbol{w}}^{(0,0)}} = 0$, and $l^{\widetilde{\boldsymbol{w}}^{(0,0)}} = n$.

with $\min(r,s) \geqslant 1$, $\xi_{\boldsymbol{p},\widetilde{\boldsymbol{w}}^{(r,s)}}$ is a normalized capacity on N given by

$$\xi_{\boldsymbol{p},\widetilde{\boldsymbol{w}}^{(r,s)}}(A) = \begin{cases} 0, & \text{if } |A| \leqslant s, \\ \frac{n}{n-(r+s)} \frac{|A|-s}{|A|} \sum_{i\in A} p_i, & \text{if } s < |A| \leqslant \left\lfloor \frac{ns}{r+s} \right\rfloor, \\ 1 - \frac{n}{n-(r+s)} \frac{n-(r+|A|)}{n-|A|} \sum_{i\notin A} p_i, & \text{if } \left\lfloor \frac{ns}{r+s} \right\rfloor < |A| < n-r, \\ 1, & \text{if } |A| \geqslant n-r. \end{cases}$$

It is worth noting that, although in this case it is also possible to obtain an expression for W (through the formula (1)), its complexity makes it of little interest.

6. Examples

In this section we will illustrate the utility of the results obtained in the previous sections through two examples. In the first of them we will show the advantages of the proposed functions over weighted means and OWA operators, while in the second one we will make a comparison with other existing methods in the literature.

Example 1. Consider the following example taken from [27] (see also [33]). Suppose that the Department of Mathematics in a Faculty of Economics offers a grant for the students accepted into the M. Sc. in Economics. Three candidates are evaluated with respect to seven subjects: Mathematics I (MatI), Mathematics II (MatII), Mathematics III (MatIII), Statistics I (StaI), Statistics II (StaII), Econometrics I (EcoI), and Econometrics II (EcoII); and the marks obtained by them (given on a scale from 0 to 10) are collected in Table 1.

Table 1: Marks of the candidates in the considered subjects.

Student	MatI	MatII	MatIII	StaI	StaII	EcoI	EcoII
A	7.9	7.8	7.7	9.8	7.5	7.6	7.4
B	7.7	7.8	7.9	5.2	8.3	8.4	8.5
C	8.2	8.4	8.5	5.2	7.7	7.8	7.9

When evaluating the candidates, the members of the committee would like to take into account the following aspects:

1. Each one of the first three subjects is considered twice as important as each one of the remaining four.
2. Minimum and maximum marks should be discarded to avoid bias.[7]

[7]Extreme marks may be due to the fact that students may have copied answers, or they may have been ill, or the same subject may have been taught by different teachers, etc. For instance, student A gets its highest grade in Statistics I whereas B and C get their lowest marks. Furthermore, these marks are also very different from those obtained in the other subjects, so it seems reasonable that they be ruled out.

The first requirement corresponds to a weighted mean type aggregation where $\boldsymbol{p} = (0.2, 0.2, 0.2, 0.1, 0.1, 0.1, 0.1)$ whereas the second one corresponds to an OWA type aggregation where $\boldsymbol{w}^{(1,1)} = (0, 2/7, 1/7, 1/7, 1/7, 2/7, 0)$ (in the case of Winsorized means) or $\widetilde{\boldsymbol{w}}^{(1,1)} = (0, 0.2, 0.2, 0.2, 0.2, 0.2, 0)$ (in the case of trimmed means).

In Table 2 we show the score given to the students by the weighted mean, the OWA operators, and the operators obtained from the Crescent Method by using $\boldsymbol{p}$, $\boldsymbol{w}^{(1,1)}$, and $\widetilde{\boldsymbol{w}}^{(1,1)}$; that is, $C_{\boldsymbol{p},\boldsymbol{w}^{(1,1)}}$ and $C_{\boldsymbol{p},\widetilde{\boldsymbol{w}}^{(1,1)}}$.

Table 2: Global evaluation of the candidates by using some aggregation operators.

Student	$M_{\boldsymbol{p}}$	$O_{\boldsymbol{w}^{(1,1)}}$	$O_{\widetilde{\boldsymbol{w}}^{(1,1)}}$	$C_{\boldsymbol{p},\boldsymbol{w}^{(1,1)}}$	$C_{\boldsymbol{p},\widetilde{\boldsymbol{w}}^{(1,1)}}$
A	7.91	7.7	7.7	7.73	$7.725\overline{6}$
B	7.72	$8.0\overline{285714}$	8.02	7.96	$7.958\overline{3}$
C	7.88	$8.0\overline{142857}$	8	8.11	8.082

As can be seen from Table 2, the winners with the weighted mean and the OWA operators are A and B, respectively. Instead, student C is the winner when $C_{\boldsymbol{p},\boldsymbol{w}^{(1,1)}}$ or $C_{\boldsymbol{p},\widetilde{\boldsymbol{w}}^{(1,1)}}$ are used. Notice that, according to the data of Table 1, the choice of student C seems the most suitable for taking into account both the importance of the subjects and the lack of bias.

As we have seen in the previous sections, the Shapley values of $C_{\boldsymbol{p},\boldsymbol{w}^{(1,1)}}$ and $C_{\boldsymbol{p},\widetilde{\boldsymbol{w}}^{(1,1)}}$ may not coincide with the weights of the vector $\boldsymbol{p}$. In fact, from expressions (1) and (3) we can get the Shapley values of these operators, which are collected in Table 3.[8]

Table 3: Shapley values of $C_{\boldsymbol{p},\boldsymbol{w}^{(1,1)}}$ and $C_{\boldsymbol{p},\widetilde{\boldsymbol{w}}^{(1,1)}}$.

Operator	ϕ_1, ϕ_2, ϕ_3	ϕ_4, ϕ_5, ϕ_6, ϕ_7
$C_{\boldsymbol{p},\boldsymbol{w}^{(1,1)}}$	$0.1\overline{809523}$	$0.1\overline{142857}$
$C_{\boldsymbol{p},\widetilde{\boldsymbol{w}}^{(1,1)}}$	$0.17\overline{396825}$	$0.11\overline{952380}$

As expected from comments made in Section 4, the Shapley values of the first three subjects are less than 0.2 (given that $0.2 > 1/7$) while the Shapley values of the remaining subjects are greater than 0.1 (given that $0.1 < 1/7$). Since the Shapley values reflect the global importance of each subject, the vector $\boldsymbol{p}$ should be chosen so that the Shapley values are 0.2 for the first three subjects and 0.1 for the remaining subjects. This can be done through Corollary 2,

[8]The values of W are $W = 5/7$ for $\boldsymbol{w}^{(1,1)}$, and $W = 64/105$ for $\widetilde{\boldsymbol{w}}^{(1,1)}$.

and we obtain the vector $\boldsymbol{p}'$ for $\boldsymbol{w}^{(1,1)}$ and $\boldsymbol{p}''$ for $\widetilde{\boldsymbol{w}}^{(1,1)}$, where

$$p'_i = \begin{cases} \frac{32}{140}, & \text{if } i = 1,2,3, \\ \frac{11}{140}, & \text{otherwise,} \end{cases} \qquad p''_i = \begin{cases} \frac{85}{343}, & \text{if } i = 1,2,3, \\ \frac{22}{343}, & \text{otherwise.} \end{cases}$$

Table 4 shows the score obtained by the candidates using the previous vectors. Note that the differences of C's scores with those of the other students have increased because C achieves his best marks in the subjects with the greatest weights (which previously did not have a Shapley value of 0.2). It is also important to note that, in this example, the results obtained through the functions $C_{\boldsymbol{p}',\boldsymbol{w}^{(1,1)}}$ and $C_{\boldsymbol{p}'',\widetilde{\boldsymbol{w}}^{(1,1)}}$ are very similar, so it seems more appropriate to use $C_{\boldsymbol{p}',\boldsymbol{w}^{(1,1)}}$ for simplicity.

Table 4: Global evaluation of the candidates by using the weighting vectors $\boldsymbol{p}'$ and $\boldsymbol{p}''$.

Student	$C_{\boldsymbol{p}',\boldsymbol{w}^{(1,1)}}$	$C_{\boldsymbol{p}'',\widetilde{\boldsymbol{w}}^{(1,1)}}$
A	7.745	$7.74\overline{714285}$
B	$7.92\overline{571428}$	7.906734...
C	$8.157\overline{857142}$	8.150612...

Example 2. Consider the following example taken from [4]. Suppose a selection committee made up of six members: three professors from distinct research fields, two other academics, and the head of department as the chair. The opinions of the committee members are weighted using the vector $\boldsymbol{p} = (2/11, 2/11, 2/11, 1/11, 1/11, 3/11)$. Besides, suppose the professors want to strengthen their research team by what they could give extreme scores in order to favor applicants conducting research in their area. To avoid this bias, an OWA type aggregation should be used in order to rule out extreme values. The OWA operator proposed for this purpose is the one associated with the vector $\boldsymbol{w} = (0, 0.25, 0.25, 0.25, 0.25, 0)$ (note that this OWA operator is also known in the literature as a trimmed mean). Table 5, given in [4], shows the scores assigned by the committee members to five applicants,[9] together with the overall score obtained through three families of functions (see [4] for more details).

It is important to note that the choice of the weighting vector $\boldsymbol{w}$ was made in order to discard extreme values (in this case the maximum and minimum values). However, as it has been pointed out in [4] and it can be seen in Table 5, none of the analyzed methods rule out extreme scores.[10]

For its part, the weighting vector $\boldsymbol{p}$ was chosen in order to reflect the importance of each committee member. But, in the framework of Choquet integral, this value is determined by means of an importance index (usually the Shapley

[9] Boldface values indicate outliers.

[10] Notice that in this example the scores of each applicant are all the same except one (the outlier), so the global score would be expected to match them. However, this is not the case (see, for instance, applicant E).

Table 5: Individual evaluations and overall score obtained by the applicants.

Applicant	Evaluations	PnTA	WOWA	Implicit
A	$(1, \mathbf{0}, 1, 1, 1, 1,)$	1	0.98	0.85
B	$(\mathbf{1}, 0.5, 0.5, 0.5, 0.5, 0.5)$	0.5	0.5	0.57
C	$(0.8, 0.8, \mathbf{0}, 0.8, 0.8, 0.8)$	0.8	0.78	0.68
D	$(0.8, 0.8, 0.8, \mathbf{0}, 0.8, 0.8)$	0.8	0.8	0.8
E	$(0.8, 0.8, 0.8, 0.8, 0.8, \mathbf{0})$	0.65	0.67	0.68

value). Given that in the PnTA method the capacities are unknown and the implicit method is not based on Choquet integrals, we only calculate the Shapley values for WOWA operators. As in the Introduction section of this paper, we consider that the quantifier Q is obtained using a linear interpolation and the Shapley values are found through the *Kappalab R package* [14]:

$$\phi_1(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}) = \phi_2(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}) = \phi_3(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}) = 0.17\overline{42}, \qquad \phi_4(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}) = \phi_5(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}) = 0.08\overline{3}, \qquad \phi_6(\mu^Q_{\boldsymbol{p},\boldsymbol{w}}) = 0.31\overline{06}.$$

Note that the values $0.17\overline{42}$ and $0.08\overline{3}$ are relative close to $2/11 = 0.\overline{18}$ and $1/11 = 0.\overline{09}$, respectively. However, the Shapley value of the sixth member of the committee differs considerably from what is desired (the error made is 13.9%). Therefore, regarding the weighting of the committee members, the behavior of the WOWA operators is not fully satisfactory either.

Now let us see what happens with the operator obtained from the Crescent Method. Notice that the weighting vector $\boldsymbol{w} = (0, 0.25, 0.25, 0.25, 0.25, 0) \in \mathcal{W}_{\Gamma}^{\vee}$ and $\mathrm{OS}_2 \leqslant O_{\boldsymbol{w}} \leqslant \mathrm{OS}_5$. Therefore, according to Corollary 1, we have $\mathrm{OS}_2 \leqslant C_{\boldsymbol{p},\boldsymbol{w}} \leqslant \mathrm{OS}_5$ for any weighting vector $\boldsymbol{p}$; that is, the operator $C_{\boldsymbol{p},\boldsymbol{w}}$ discards the maximum and minimum values, which was the goal for which the vector $\boldsymbol{w}$ was chosen. Hence, in this example, when $\boldsymbol{x}$ is a vector where all the scores except one are the same (see Table 5), we get $C_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \mathrm{OS}_2(\boldsymbol{x}) = \mathrm{OS}_5(\boldsymbol{x})$. Therefore, the scores obtained by applicants A, B, C, D, and E are 1, 0.5, 0.8, 0.8, and 0.8, respectively.

In relation to the weighting given to the committee members, the Shapley values of $C_{\boldsymbol{p},\boldsymbol{w}}$ can be obtained from expressions (1) and (3):

$$\phi_1(\xi_{\boldsymbol{p},\boldsymbol{w}}) = \phi_2(\xi_{\boldsymbol{p},\boldsymbol{w}}) = \phi_3(\xi_{\boldsymbol{p},\boldsymbol{w}}) = 0.17\overline{42}, \qquad \phi_4(\xi_{\boldsymbol{p},\boldsymbol{w}}) = \phi_5(\xi_{\boldsymbol{p},\boldsymbol{w}}) = 0.12\overline{87}, \qquad \phi_6(\xi_{\boldsymbol{p},\boldsymbol{w}}) = 0.21\overline{96}.$$

Notice that the Shapley values of the last three committee members are not what would be desired. However, in this case, given that we know the expressions that relate the Shapley values to the weighting vectors $\boldsymbol{p}$ (Corollary 2), we can obtain a weighting vector $\boldsymbol{p}'$ so that the Shapley values of $C_{\boldsymbol{p}',\boldsymbol{w}}$ are $(2/11, 2/11, 2/11, 1/11, 1/11, 3/11)$. This vector is $\boldsymbol{p}' = (13/66, 13/66, 13/66, 1/66, 1/66, 25/66)$. So, the operator $C_{\boldsymbol{p}',\boldsymbol{w}}$ meets the set requirements: it allows to discard extreme values and gives each member of the committee the desired weight.

7. Conclusion

There are several methods in the literature that allow us to build functions from two weighting vectors, $\boldsymbol{p}$ and $\boldsymbol{w}$, where the first one allows weighting of the information sources, while the second one is used for an OWA type aggregation. However, on some occasions, the vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ do not have the desired role in the obtained functions. Hence, it is essential to know the relationship between the weights that the function gives to the information sources and the weighting vector $\boldsymbol{p}$, or make sure that the function discards extreme values when the vector $\boldsymbol{w}$ corresponds to a trimmed or Winsorized mean.

In this paper we have analyzed the conjunctive/disjunctive character of $C_{\boldsymbol{p},\boldsymbol{w}}$ (that is, the Choquet integral with respect to the capacity $\xi_{\boldsymbol{p},\boldsymbol{w}}$), and we have seen that $C_{\boldsymbol{p},\boldsymbol{w}}$ maintains the k-conjunctive/k-disjunctive character of $O_{\boldsymbol{w}}$. Besides, we have also given closed-form expressions of several indices such as k-conjunctiveness and k-disjunctiveness indices, the veto and favor indices, and the Shapley values. The usefulness of the results obtained has been illustrated through two examples in Section 6. In the first example we have shown the advantages of the proposed functions over weighted means and OWA operators while in Example 2 we have made a comparison with other methods proposed in the literature. As it has been seen in this last example, the functions studied in this paper allow both to discard extreme values and to give the information sources the desired weight, which is not the case with the remaining methods analyzed in the example.

Acknowledgements

The author is grateful to two anonymous referees for their insightful comments that helped improve the paper.

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