Modeling rationality in a linguistic framework

José Luis García-Lapresta and Luis Carlos Meneses

PRESAD Research Group, Dep. of Applied Economics, University of Valladolid, Avenida Valle de Esgueva 6, 47011 Valladolid, Spain

Abstract

In classical decision theory there exists a large class of rationality models which try to capture different kinds of behavior when individuals compare by pairs a set of alternatives. All these models assume that decision makers have dichotomous preferences. However, in real decisions individuals feel different degrees of preference. In this paper we have checked the mentioned models in a real case where different kinds of linguistic preferences are allowed. After the empirical analysis, the main conclusion is that the fulfillment of rational conditions decreases when individuals have non-extreme preferences. Based on the obtained empirical evidences, we propose some classes of transitivity conditions in the framework of linguistic preferences.

Keywords: rationality, linguistic preferences, transitivity, monotonic operators.

1 Introduction

Classical models of rational behavior only consider dichotomous assessments among pairs of alternatives (preference or not preference). Thus, intensities of preference are not allowed and all the modalities of preference are identified. Since individuals usually feel different intensities of preference among alternatives, other models are necessary. In this sense, fuzzy preferences are a relevant tool for modeling preference intensities (see Bezdek et al. [1], Nurmi [33], Tanino [43] and De Baets and Fodor [5], among others). Fuzzy preferences represent intensities of preference by means of numerical values within the unit interval.

In spite of the fact that fuzzy preferences have been widely used in decision making to allow individuals to express different modalities of preference, in

Email addresses: `lapresta@eco.uva.es` (José Luis García-Lapresta), `lmeneses@eco.uva.es` (Luis Carlos Meneses).

some occasions individuals have difficulties declaring intensities of preference in a quantitative manner. In those cases, the use of linguistic labels (or finite scales as in Mayor and Torrens [32]) to grade preferences could be more appropriate to capture the lack of precision in human behavior (see Zadeh [47–49], Herrera et al. [26,27], Herrera and Herrera-Viedma [25], García-Lapresta [17] and García-Lapresta et al. [19], among others).

Linguistic preference modeling analyzes how to manage human preferences when they are qualitative rather than quantitative. For our purposes, we can distinguish two different approaches to linguistic preference modeling. The first one requires a semantics to represent linguistic labels by means of appropriate membership functions, usually with triangular or trapezoidal shapes (see, for instance, Herrera and Herrera-Viedma [25]). The second approach does not need any representation of linguistic terms and it only works in a symbolic way. In this paper we follow the second approach to analyze the fulfillment of some rationality conditions related to those transitivity properties introduced by Díaz et al. [8]. These conditions are based on linear orders, complete preorders, transitivity of preference and indifference, and triple-acyclicity. We have checked these models of rational behavior in some real cases where individuals could show their preferences by means of linguistic terms.

Our empirical research is based on the data included in García-Lapresta and Meneses [20,21], where groups of 50 students compared by pairs the alternatives of 3 different sets of 6 alternatives by means of 4 modalities or preference (totally, highly, rather and slightly); in absence of preference they declare indifference. We analyze the fulfillment of the above mentioned classical models of rational behavior with respect to the obtained data but taking into account the modalities of preference used by the individuals.

Taking into account the empirical data, we conclude that transitivity of crisp preferences is only fulfilled whenever individuals have extreme preferences among alternatives. Clearly, classical models of rational behavior are not appropriate for modeling human preferences. This is the reason why we have introduced the class of F-transitive preference relations for appropriate monotonic operators F, in the context of linguistic preferences.

The paper is organized as follows. Section 2 is devoted to introduce terminology, the classical models of rational behavior we check in the empirical analysis and linguistic preferences. In Section 3 we include the empirical research. Section 4 contains a proposal of rationality models within the framework of linguistic preferences based on monotonic operators. Finally, Section 5 includes some conclusions.

2 Preliminaries

Let S, T be two binary relations on a set A. Δ is the *identity relation* of A $[a \, \Delta \, b \;\Leftrightarrow\; a = b]$; S^{-1} is the *inverse relation* of S $[a \, S^{-1} \, b \;\Leftrightarrow\; b \, S \, a]$; S^c is the *complement relation* of S $[a \, S^c \, b \;\Leftrightarrow\; \text{not } (a \, S \, b)]$; $S \cap T$ is the *intersection relation* of S and T $[a \, (S \cap T) \, b \;\Leftrightarrow\; (a \, S \, b \text{ and } a \, T \, b)]$; $S \cup T$ is the *union relation* of S and T $[a \, (S \cup T) \, b \;\Leftrightarrow\; (a \, S \, b \text{ or } a \, T \, b)]$; $S \circ T$ is the *composition relation* of S and T $[a \, (S \circ T) \, b \;\Leftrightarrow\; \exists \, c \in A \;\; (a \, S \, c \text{ and } c \, T \, b)]$.

A way of introducing classical preference and indifference concepts is taking the (strong) *preference* as primitive notion, through an ordinary binary relation P on a set of alternatives A, where $a \, P \, b$ means "a is preferred to b" or "a is better than b". A basic assumption for P is *asymmetry*: $P \cap P^{-1} = \emptyset$. In this case the *indifference relation*, I, is defined by absence of preference, i.e., a is indifferent to b when neither a is preferred to b nor b is preferred to a: $I = (P \cup P^{-1})^c = P^c \cap (P^{-1})^c$. Then, the *weak preference* relation is defined by $P \cup I = (P^{-1})^c$. We note that from the asymmetry of P, we obtain that I is reflexive $[\forall a \in A \quad a \, I \, a]$ and symmetric $[\forall a, b \in A \quad a \, I \, b \;\Rightarrow\; b \, I \, a]$; and the completeness of $P \cup I$ $[\forall a, b \in A \quad a \, (P \cup I) \, b \text{ or } b \, (P \cup I) \, a]$. Thus, for all $a, b \in A$, one and only one of the following facts holds: $a \, P \, b$ or $b \, P \, a$ or $a \, I \, b$.

2.1 A short review on rationality, transitivity and consistency

Consistency in decision making has different meanings. Depending on the framework and the way in which individuals show their preferences, consistency and rationality can be understood in a different manner.

First we pay attention to crisp preference relations, where individuals show in a dichotomous way whether an alternative is preferred to another one or they are indifferent. In this context, it is not possible to show intensities of preference among alternatives.

2.1.1 Crisp preference relations

Surely, the most classical notion of consistency is related to the existence of a utility function. So, if an individual assigns a value to each alternative through a utility function $u : A \longrightarrow \mathbb{R}$, we can define a (strong) preference relation in the following way: $a \, P \, b$ if $u(a) > u(b)$. It is clear that P is a *complete preorder* (or *weak order*): P is asymmetric and transitive, and the indifference relation I, defined by $a \, I \, b$ if $u(a) = u(b)$, is reflexive, symmetric and transitive, i.e., an equivalence relation.

Within the above classic consistency model (complete preorders), the hierarchy of satisfied rationality properties can be summarized as follows (see, for instance, García-Lapresta and Rodríguez-Palmero [24]):

(1) Comparing an alternative with itself.
   (a) Irreflexivity of P: not $a\,P\,a$, for every $a \in A$.
   (b) Reflexivity of I: $a\,I\,a$, for every $a \in A$.
(2) Comparing two alternatives.
   (a) Asymmetry of P: if $a\,P\,b$, then not $b\,P\,a$, for all $a, b \in A$.
   (b) Symmetry of I: if $a\,I\,b$, then $b\,I\,a$, for all $a, b \in A$.
   (c) Trichotomy: for all $a, b \in A$, one and only one of the following facts holds: $a\,P\,b$ or $b\,P\,a$ or $a\,I\,b$.
(3) Comparing three alternatives by pairs.
   (a) P-transitivity: $P \circ P \subseteq P$, i.e., if $a\,P\,b$ and $b\,P\,c$, then $a\,P\,c$, for all $a, b, c \in A$.
   (b) I-transitivity: $I \circ I \subseteq I$, i.e., if $a\,I\,b$ and $b\,I\,c$, then $a\,I\,c$, for all $a, b, c \in A$.
   (c) PI-transitivity: $P \circ I \subseteq P$, i.e., if $a\,P\,b$ and $b\,I\,c$, then $a\,P\,c$, for all $a, b, c \in A$.
   (d) IP-transitivity: $I \circ P \subseteq P$, i.e., if $a\,I\,b$ and $b\,P\,c$, then $a\,P\,c$, for all $a, b, c \in A$.

This classic model has been violated in numerous empirical cases with respect to transitivity. Thus, Luce [30] proposed to eliminate the transitivity of the indifference relation and he introduced the class of *semiorders* (preference relations satisfying $P \circ P \circ I \subseteq P$ and $P \circ I \circ P \subseteq P$). Similarly, Fishburn [14] introduced the class of *interval orders* (preference relations satisfying $P \circ I \circ P \subseteq P$), and Sen [36] introduced the class of *quasi-transitive* preference relations (preference relations satisfying P-transitivity).

The transitivity of the strong preference relation has been also questioned (see May [31], Tversky [44] and Van Acker [45], among others). In this way, Sen [37] introduced a weaker condition than P-transitivity, *acyclicity*: if $a_1\,P\,a_2$, $a_2\,P\,a_3$, …, $a_{r-1}\,P\,a_r$, then not $a_r\,P\,a_1$, for all $a_1, \ldots, a_r \in A$. The weakest rationality condition involving three alternatives, *triple-acyclicity*, appears in Sen [38, p. 62], Suzumura [39, p. 8] and Schwartz [35, p. 56]: $P \circ P \subseteq P \cup I$, i.e., $(a\,P\,b$ and $b\,P\,c) \Rightarrow (a\,P\,c$ or $a\,I\,c)$, for all $a, b, c \in A$.

Among the wide variety of rationality models within the classical approach to preference modeling, we have considered triple-acyclicity, transitivity of preference and indifference relations, complete preorders and *linear orders* (complete preorders satisfying $I = \Delta$). For other models, see Roubens and Vincke [34] and García-Lapresta and Rodríguez-Palmero [24], among others.

In Figure 1 we show the implications among the considered classical models of rationality (see, for instance, García-Lapresta and Rodríguez-Palmero [24]).

Fig. 1. Relationships among rationality models

$$\begin{tikzcd} & \text{Linear order} \arrow[d] & \\ & \text{Complete preorder} \arrow[dl] \arrow[dr] & \\ P\text{-transitive} \arrow[d] & & I\text{-transitive} \\ \text{Triple-acyclic} & & \end{tikzcd}$$

2.1.2 Fuzzy preference relations

Fuzzy and reciprocal preference relations generalize crisp preference relations by allowing individuals to show different degrees of preference among alternatives within the unit interval (see Nurmi [33], Tanino [43], Fodor and Roubens [16], De Baets and Fodor [5], García-Lapresta and Llamazares [18], De Baets and De Meyer [3], and De Baets et al. [4], among others).

A *reciprocal relation* on A is a mapping $R : A \times A \longrightarrow [0, 1]$, where $R(a, b)$ is the intensity of preference between a and b, and R satisfies the *reciprocity* condition: $R(a, b) + R(b, a) = 1$ for all $a, b \in A$. Notice that $R(a, a) = 0.5$ for every $a \in A$. On the other hand, if $R(a, b) \in \{0, 0.5, 1\}$ for all $a, b \in A$, then reciprocity becomes asymmetry and R can be seen as a crisp preference relation.

Each reciprocal relation has associated a crisp preference relation P_R, defined by $a\, P_R\, b$ if $R(a, b) > 0.5$. The indifference relation I_R is defined as $a\, I_R\, b$ if $R(a, b) = 0.5$. Clearly, P_R is asymmetric, hence irreflexive; I_R is reflexive and symmetric; and for all $a, b \in A$, one and only one of the following facts holds: $a\, P_R\, b$ or $b\, P_R\, a$ or $a\, I_R\, b$.

Reciprocal relations capture a natural situation: if an individual compares two alternatives and one alternative is preferred to the other, then this individual can show the intensity of preference between the best and the worst alternatives. It is important to note that individuals do not need to compare the alternatives twice: if a is preferred to b, i.e., $R(a, b) > 0.5$, then

$R(b,a) = 1 - R(a,b) < 0.5$. This fact and the previous properties are not satisfied in the general class of fuzzy preference relations $R : A \times A \longrightarrow [0,1]$ where reciprocity is not required or is weakened to the condition $R(a,b) + R(b,a) \geq 1$ for all $a, b \in A$.

Transitivity has been generalized to the fuzzy framework in different ways (see, for instance, Dubois and Prade [13], Jain [29], Dasgupta and Deb [2], Switalski [40–42], Herrera-Viedma et al. [28], García-Lapresta and Meneses [20,21] and García-Lapresta and Montero [23]). Among these transitivity conditions, one of the most frequent fuzzy transitivity conditions is that of T-transitivity, where $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a t-norm: $R(a,b) \geq T(R(a,c), R(c,b))$ for all $a, b, c \in A$ such that $R(a,c), R(c,b) > 0.5$. It is important to note that in 1973 Fishburn [15] provided some extensions of transitivity within the probabilistic choice framework. These extensions can be considered as precursors of some fuzzy transitivities in the field of reciprocal relations. More recently, De Baets and De Meyer [3] and De Baets et al. [4] have proposed other interesting generalizations of transitivity in the field of reciprocal relations. Other contributions related to transitivity in the fuzzy context are De Baets et al. [7], De Baets and Van de Walle [6], Van de Walle et al. [46], Díaz et al. [11,12,9,10].

2.2 Linguistic preferences

Let A be a finite set of alternatives and $\mathcal{L}_n = \{l_0, l_1, \ldots, l_{2n}\}$ be a set of linguistic labels, where $n \geq 1$, ranked by a linear order: $l_0 < l_1 < \cdots < l_{2n}$. The intermediate label l_n represents indifference, and the rest of labels are defined around it symmetrically. We denote $\mathcal{L}_n^+ = \{l_{n+1}, \ldots, l_{2n}\}$.

In the framework of fuzzy preference relations, reciprocity is equivalent to: $R(a,b) = k$ and $R(b,a) = k'$ imply that $k + k' = 1 = \max[0,1]$, for all $a, b \in X$ and $k, k' \in [0,1]$. Taking into account this idea, we can extend reciprocity to the framework of linguistic preferences in the following sense: $R(a,b) = l_k$ and $R(b,a) = l_{k'}$ imply that $k + k' = 2n = \max\{0, 1, \ldots, 2n\}$, for all $a, b \in X$ and $k, k' \in \{0, 1, \ldots, 2n\}$. But this is equivalent to $R(a,b) = l_k$ implies $R(b,a) = l_{2n-k}$, for all $a, b \in X$ and $k \in \{0, 1, \ldots, 2n\}$.

Definition 1 *A* linguistic preference relation on *A* based on $\mathcal{L}_n$ *is a mapping* $R : A \times A \longrightarrow \mathcal{L}_n$ *satisfying the* reciprocity *condition:*

$$R(a,b) = l_k \quad \Rightarrow \quad R(b,a) = l_{2n-k}$$

for all $a, b \in A$ *and* $k \in \{0, 1, \ldots, 2n\}$.

We denote by $\mathcal{L}_n(A)$ the set of the linguistic preference relations on A based on $\mathcal{L}_n$.

The meaning of the labels can be summarized in this way: $R(a,b) = l_{2n}$, if a is totally preferred to b; $l_n < R(a,b) < l_{2n}$, if a is somewhat preferred to b; $R(a,b) = l_n$, if a is indifferent to b; $l_0 < R(a,b) < l_n$, if b is somewhat preferred to a; $R(a,b) = l_0$, if b is totally preferred to a. Since l_k and l_{2n-k} represent the same modality of preference, used in a symmetric way, in $\mathcal{L}_n$ there are $n+1$ possibilities for declaring opinions between pairs of alternatives: n kinds for preference and 1 for indifference.

We note that $\mathcal{L}_1 = \{l_0, l_1, l_2\}$ consists on the classical preference model. When other modalities of preference are allowed, we should use $\mathcal{L}_n$ for $n > 1$. In fact, the sets of linguistic labels $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ have been widely used in the literature. In the empirical research we use $\mathcal{L}_4$.

Given $R \in \mathcal{L}_n(A)$, let P_R the ordinary binary relation on A defined by $a\, P_R\, b$ if $R(a,b) > l_n$. Clearly P_R is asymmetric, hence an ordinary preference relation. We say that P_R is the *ordinary preference relation associated with* R. The *ordinary indifference relation associated with* R, I_R, is just the indifference relation associated with P_R: $a\, I_R\, b$ if $R(a,b) = l_n$.

3 The empirical analysis

We have considered the data included in García-Lapresta and Meneses [20]. In that paper two sets of alternatives were analyzed with respect to the fulfillment of several classes of fuzzy transitivity. A group of 50 students compared by pairs 6 different trips. We note that in Spain before students finish their degree, they usually go together for a trip. Previously they have to decide the trip among different possibilities. We asked them about a first set of alternatives with 6 destinations: China, Egypt, Prague-Budapest, Italy, Cuba and Russia. In a second stage, the same group of students compared the same trips but now taking into account the final price of these trips (in euros). This is the second set of alternatives. In García-Lapresta and Meneses [21] we analyzed individual and collective rationality taking into account the preferences of a group of students with respect to 6 degrees just in the moment they registered in their first year in our Faculty. We have considered the data included in that paper for our present analysis of this third set of alternatives.

The sets of alternatives involved in our empirical research are:

$X_1 = \{$China, Egypt, Prague-Budapest, Italy, Cuba, Russia$\}$.

$X_2 = \{$(China, 630), (Egypt, 450), (Prague-Budapest, 270), (Italy, 150), (Cuba, 540), (Russia, 330)}.

X_3 = {Business Administration and Management (5 years), Business Administration (3 years), Law (5 years), Business Administration, Management and Law (6 years), Labor Relations (3 years), Economics (5 years)}.

In all the cases, students compare by pairs the feasible alternatives by means of 4 modalities of linguistic preference (totally, highly, rather, slightly) and indifference (see Table 1).

Table 1
The semantics of $\mathcal{L}_4$

$R(a,b)$	$R(b,a)$	Meaning
l_8	l_0	a is totally preferred to b
l_7	l_1	a is highly preferred to b
l_6	l_2	a is rather preferred to b
l_5	l_3	a is slightly preferred to b
l_4	l_4	a is indifferent to b
l_3	l_5	b is slightly preferred to a
l_2	l_6	b is rather preferred to a
l_1	l_7	b is highly preferred to a
l_0	l_8	b is totally preferred to a

We note that $a\, P_R\, b$ if $R(a,b) > l_4$; $a\, I_R\, b$ if $R(a,b) = l_4$; and $b\, P_R\, a$ if $R(a,b) < l_4$.

In Table 2 we show the total percentages of individuals who fulfill the rationality models appearing in Figure 1. According to the implications showed in Figure 1, these percentages decrease when the conditions are more restrictive. We note that these percentages are higher in the set X_3 than in X_2, excepting for I-transitivity, and in X_2 than in X_1. This fact could be explained by a more accurate information and interest for the students. It is important to note that no property has been fulfilled by all the individuals. Obviously, the smaller percentages of unfulfillment correspond to triple-acyclicity: in the three sets, 10% of individuals do not satisfy this property. Moreover, according to the empirical data, linear orders, even complete preorders, can not be consider as realistic models of rational behavior.

Now we deal with the relative fulfillment of some rationality models taking into account the different modalities of preference the students showed in the pairwise comparisons of alternatives. This relative fulfillment means the percent-

Table 2
Percentages of individuals who fulfill the rationality models

	T-A	I-T	P-T	C.P.	L.O.
X_1	90%	64%	62%	46%	12%
X_2	90%	82%	76%	62%	28%
X_3	90%	76%	82%	62%	36%

ages of cases where the corresponding properties are satisfied with respect to all the possible cases. For instance, P_R is P-transitive if $(a\,P_R\,b$ and $b\,P_R\,c) \Rightarrow a\,P_R\,c$, for all $a,b,c \in A$. However, it is not the same to violate the above implication in one triplet than in almost all possible cases. The property is not satisfied in both situations, but in a different degree.

Table 3
Triple-acyclicity and P-transitivity in X_2

$R(a,b)$ / $R(b,c)$	$R(a,c)$: l_0,l_1,l_2,l_3	$R(a,c)$: l_4	$R(a,c)$: l_5,l_6,l_7,l_8
l_5,l_5	11.63%	9.30%	79.07%
l_5,l_6	5.11%	2.19%	92.70%
l_5,l_7	2.02%	3.03%	94.95%
l_5,l_8	3.09%	0.00%	96.91%
l_6,l_6	1.08%	2.15%	96.77%
l_6,l_7	0.00%	2.44%	97.56%
l_6,l_8	1.30%	0.00%	98.70%
l_7,l_7	0.00%	0.00%	100%
l_7,l_8	0.00%	0.00%	100%
l_8,l_8	0.00%	0.00%	100%
Total	3.03%	2.28%	94.69%

In Table 3 we include the percentages of relative fulfillment in X_2 by rows. For instance, the second row shows how is $R(a,c)$ whenever $R(a,b) = l_5$ and $R(b,c) = l_6$, or $R(a,b) = l_6$ and $R(b,c) = l_5$. The first column contains the relative unfulfillment of triple-acyclicity ($R(a,c) < l_4$); the sum of the second and third columns corresponds to the relative fulfillment of triple-acyclicity ($R(a,c) \geq l_4$); the sum of the first and second columns corresponds to the relative unfulfillment of P-transitivity ($R(a,c) \leq l_4$); and the third column, the relative fulfillment of P-transitivity ($R(a,c) > l_4$). All these percentages are relative to the number of cases where individuals can fulfill the specific condition. For instance, in column 1 row 1, 11.63% is the percentage of cases where individuals declare that a is slightly preferred to b ($R(a, b) = l_5$), b is slightly preferred to c ($R(b, c) = l_5$) and c is somewhat preferred to a ($R(a, c) < l_4$, i.e., $R(c, a) > l_4$). The last row includes the global information without paying attention to preference intensities $R(a, b)$ and $R(b, c)$. Notice that only the 3.03% of the triplets unfulfill triple-acyclicity and 5.31% of the triplets unfulfill P-transitivity. It is important to note that the fulfillment's percentages increase when individuals show higher intensities of preference in almost all the cases.

In the Appendix we include the data for X_1 and X_3.

In García-Lapresta and Meneses [22] other classical rationality models have been analyzed: semiorders, interval orders and semitransitive relations. We note that the fulfillment's percentages of these models are very close to that of P-transitivity.

4 F-transitivity

In ordinary preferences, transitivity is the crucial point for modeling rationality. The same happens in fuzzy preferences, but there exist a wide class of transitivity notions for extending this property to the fuzzy framework (see Dasgupta and Deb [2], Switalski [40–42], García-Lapresta and Meneses [20,21], De Baets and De Meyer [3], and De Baets et al. [4], among others).

According to the empirical analysis, a clear conclusion is that percentages of unfulfillment of the considered rationality models decrease whenever the preference intensities increase. In fact, the case of extreme preference (l_8) hardly produces inconsistencies. In this sense, we can state that the classic models of rational behavior are appropriate for extreme preferences, but not for slight preferences. Thus, it is necessary to consider new models of rational behavior which take into account modalities of preference. In order to define the concept of transitivity in the framework of linguistic preferences, we could consider the idea of t-norm on a finite scale (see Mayor and Torrens [32]). However, for this purpose, associativity and boundary conditions are not essential. Without these properties, we are going to work only with the commutative and monotonicity conditions by means of the class of monotonic operators, a wider class than the class of t-norms, which allows us to define transitivity in a very general way within the linguistic framework.

Definition 2 *A* monotonic operator *on $\mathcal{L}_n^+$ is a mapping $F : \mathcal{L}_n^+ \times \mathcal{L}_n^+ \longrightarrow \mathcal{L}_n$ satisfying the following conditions:*

(1) Commutativity*: $F(l_i, l_j) = F(l_j, l_i)$, for all $l_i, l_j \in \mathcal{L}_n^+$.*
(2) Monotonicity*: $F(l_i, l_j) \leq F(l_i, l_k)$, for all $l_i, l_j, l_k \in \mathcal{L}_n^+$ such that $l_j \leq l_k$.*
(3) *$F(l_{2n}, l_{2n}) > l_n$.*

Notice that the last condition is weaker than the usual boundary condition (adapted to our linguistic framework) $F(l_{2n}, l_{2n}) = l_{2n}$. On the other hand, it is important to note that associativity

$$F(l_i, F(l_j, l_k)) = F(F(l_i, l_j), l_k),$$

for all $l_i, l_j, l_k \in \mathcal{L}_n^+$, has not sense in our definition of monotonic operator. Since the codomain of these operators is $\mathcal{L}_n$, if $F(l_j, l_k) \leq l_n$ or $F(l_i, l_j) \leq l_n$, then the above equality is meaningless.

Examples of monotonic operators are the minimum and the maximum operators.

Definition 3 *Let F be a monotonic operator on $\mathcal{L}_n^+$ and let $R \in \mathcal{L}_n(A)$. R is F-*transitive *if $R(a, b) \geq F(R(a, c), R(c, b))$ for all $a, b, c \in A$ such that $R(a, c), R(c, b) \in \mathcal{L}_n^+$.*

In the next Proposition we show some interesting facts for two specific constant monotonic operators, those which satisfy $F(l_{n+1}, l_{n+1}) \in \mathcal{L}_n^+$, and one for the case of $n = 1$ (the classical preference model).

Proposition 4 *Let F be a monotonic operator on $\mathcal{L}_n^+$ and let $R \in \mathcal{L}_n(A)$. The following statements hold:*

(1) *If $F(l_i, l_j) = l_n$ for all $l_i, l_j \in \mathcal{L}_n^+$, then R is F-transitive if and only if P_R is triple-acyclic.*
(2) *If $F(l_i, l_j) = l_{n+1}$ for all $l_i, l_j \in \mathcal{L}_n^+$, then R is F-transitive if and only if P_R is P-transitive.*
(3) *If $F(l_{n+1}, l_{n+1}) \geq l_{n+1}$ and R is F-transitive, then P_R is P-transitive.*
(4) *For $n = 1$, R is F-transitive if and only if P_R is P-transitive.*

PROOF.

(1) Suppose R is F-transitive, $a\, P_R\, b$ and $b\, P_R\, c$. Since $R(a, b), R(b, c) \in \mathcal{L}_n^+$, we have $R(a, c) \geq F(R(a, b), R(b, c)) = l_n$ and, consequently, $a\, P_R\, c$ or $a\, I_R\, c$.
Conversely, suppose P_R is triple-acyclic and $R(a, b), R(b, c) \in \mathcal{L}_n^+$. Then, $a\, P_R\, b$ and $b\, P_R\, c$. Therefore, $a\, P_R\, c$ or $a\, I_R\, c$, i.e., $R(a, c) \geq l_n = F(R(a, b), R(b, c))$.
(2) Suppose R is F-transitive, $a\, P_R\, b$ and $b\, P_R\, c$. Since $R(a, b), R(b, c) \in \mathcal{L}_n^+$, we have $R(a, c) \geq F(R(a, b), R(b, c)) = l_{n+1}$ and, consequently, $a\, P_R\, c$.
Conversely, suppose P_R is P-transitive and $R(a, b), R(b, c) \in \mathcal{L}_n^+$. Since $a\, P_R\, b$ and $b\, P_R\, c$, we have $a\, P_R\, c$, i.e., $R(a, c) \geq l_{n+1} = F(R(a, b), R(b, c))$.

(3) Suppose R is F-transitive, $a\, P_R\, b$ and $b\, P_R\, c$. Since $R(a,b), R(b,c) \in \mathcal{L}_n^+$, then we have $R(a,c) \geq F(R(a,b), R(b,c)) \geq F(l_{n+1}, l_{n+1}) \geq l_{n+1}$ and, consequently, $a\, P_R\, c$.

(4) Suppose R is F-transitive, $a\, P_R\, b$ and $b\, P_R\, c$. Since $R(a,b), R(b,c) \in \mathcal{L}_1^+$, we have $R(a,c) \geq F(R(a,b), R(b,c)) \geq F(l_2, l_2) > l_1$ and, consequently, $a\, P_R\, c$.

Conversely, suppose P_R is P-transitive and $R(a,b), R(b,c) \in \mathcal{L}_1^+ = \{l_2\}$. Then, $a\, P_R\, b$ and $b\, P_R\, c$. Therefore, $a\, P_R\, c$, i.e., $R(a,c) = l_2 \geq F(R(a,b), R(b,c))$.

Now we are interested in searching some classes of monotonic operators which could be suitable for modeling the rational behavior through the corresponding linguistic transitivities. For this task we will consider the empirical research provided in the previous section.

4.1 Some examples

Example 5 A well known rationality model in the framework of fuzzy preferences is that of *min-transitivity*. This model can be easily defined in $\mathcal{L}_n^+$ through $F_1(l_i, l_j) = \min\{l_i, l_j\} = l_{\min\{i,j\}}$:

$$R(a,b) \geq \min\{R(a,c), R(c,b)\}$$

whenever $R(a,c), R(c,b) \in \mathcal{L}_n^+$.

Since $F_1(l_{n+1}, l_{n+1}) = l_{n+1}$, by Proposition 4, P_R is P-transitive for every $R \in \mathcal{L}_n(A)$ satisfying F_1-transitivity.

Example 6 We now consider the monotonic operators $F_k : \mathcal{L}_n^+ \times \mathcal{L}_n^+ \longrightarrow \mathcal{L}_n$, $k = 2, 3, 4$, defined by

$$F_2(l_i, l_j) = \max\{l_{i+j-2n}, l_{n+1}\},$$

$$F_3(l_i, l_j) = \max\{l_{i+j-2n}, l_n\},$$

$$F_4(l_i, l_j) = l_{i+j-2n}.$$

Notice that they have some similarities with the Łukasiewicz t-norm [1]. On the other hand, we have

$$F_1(l_i, l_j) \geq F_2(l_i, l_j) \geq F_3(l_i, l_j) \geq F_4(l_i, l_j),$$

for all $l_i, l_j \in \mathcal{L}_n^+$.

[1] The Łukasiewicz t-norm is defined on $[0,1]^2$ through $T_L(x,y) = \max\{x+y-1, 0\}$.

In Table 4 we show these monotonic operators for $n = 4$.

Table 4
F_2, F_3, F_4 monotonic operators on $\mathcal{L}_4^+$

F_2	l_5	l_6	l_7	l_8
l_5	l_5	l_5	l_5	l_5
l_6	l_5	l_5	l_5	l_6
l_7	l_5	l_5	l_6	l_7
l_8	l_5	l_6	l_7	l_8

F_3	l_5	l_6	l_7	l_8
l_5	l_4	l_4	l_4	l_5
l_6	l_4	l_4	l_5	l_6
l_7	l_4	l_5	l_6	l_7
l_8	l_5	l_6	l_7	l_8

F_4	l_5	l_6	l_7	l_8
l_5	l_2	l_3	l_4	l_5
l_6	l_3	l_4	l_5	l_6
l_7	l_4	l_5	l_6	l_7
l_8	l_5	l_6	l_7	l_8

According to Proposition 4, P_R is P-transitive for every $R \in \mathcal{L}_n(A)$ satisfying F_3-transitivity; and P_R is triple-acyclic for every $R \in \mathcal{L}_n(A)$ satisfying F_1-transitivity.

Clearly, F_4 has a different behavior depending on if it aggregates high or low intensities of preference. For instance, if R is F_4-transitive, $R(a,b) = l_6$ and $R(b,c) = l_7$, then $R(a,c)$ should be at least l_5. If $R(a,b) = l_7$ and $R(b,c) = l_8$, then $R(a,c)$ should be at least l_7. Even more, if $R(a,b) = l_5$ and $R(b,c) = l_6$, then $R(a,c)$ should only be at least l_3. Thus, F_4-transitivity does not guarantees triple-acyclicity of P_R.

In Table 5 we show the percentages of individuals who fulfill F_k-transitivity for $k = 1,2,3,4$ in the three sets of alternatives.

Table 5
Percentages of individuals who fulfill F_k-transitivity for $k = 1,2,3,4$

	F_1	F_2	F_3	F_4
X_1	52%	60%	80%	84%
X_2	48%	66%	78%	82%
X_3	54%	82%	86%	88%

It is important to note the low fulfillment of F_1-transitivity in all the cases. Thus, taking into account our empirical data, the well known rationality condition of min-transitivity (translated to the linguistic framework) cannot be considered as a realistic model of rational behavior.

F_1 and F_2 transitivities ensure P-transitivity to the associated ordinary preference relations. Notice that the only differences between F_1 and F_2 are in $\{l_6, l_7\}$, being $F_2(l_i, l_j) = l_{p-1}$ and $F_1(l_i, l_j) = l_p$ for all $i, j \in \{6, 7\}$. Obviously, F_2-transitivity is fulfilled in a bigger degree than F_1-transitivity; then F_2-transitivity is preferable to F_1-transitivity for modeling rational behavior.

If we compare F_2 and F_3, the only differences appear in low preferences:

$F_3(l_i, l_j) = l_{p-1}$ and $F_2(l_i, l_j) = l_p$ whenever $i + j < 3n + 1$. Consequently, F_3-transitivity does not ensure P-transitivity to the associated ordinary preference relations, but only triple-acyclicity.

Since there exist clear differences between the fulfillment of F_3-transitivity and F_2-transitivity, it seems that F_3-transitivity is a more reasonable coherence model than F_2-transitivity.

If we now compare F_3 and F_4, again the only differences appear in low preferences. So, for $n = 4$, F_3 and F_4 coincides excepting in the cases $F_4(l_5, l_5) = l_2 < l_4 = F_3(l_5, l_5)$ and $F_4(l_5, l_6) = F_4(l_6, l_5) = l_3 < l_4 = F_3(l_5, l_6) = F_3(l_6, l_5)$. Clearly, the weaker model of F_4-transitivity is more realistic for modeling rationality than the previous ones. However, F_4-transitivity does not ensure triple-acyclicity to the associated ordinary preference relations.

4.2 Our proposal

Taking into account the previous examples and Table 3, we can establish that F-transitivity should be defined through monotonic operators whose behavior be different depending on the strong of preferences they aggregate. Clearly, for every $(l_i, l_j) \in \mathcal{L}_n^+ \times \mathcal{L}_n^+$ we have $i + j \in [2n + 2, 4n] \cap \mathbb{N}$. Then, the middle point in this interval of positive integers is $3n + 1$. So, we differentiate in $\mathcal{L}_n^+ \times \mathcal{L}_n^+$ three disjoint subsets:

(1) Low preferences: $L = \{(l_i, l_j) \in \mathcal{L}_n^+ \times \mathcal{L}_n^+ \mid i + j < 3n + 1\}$.
(2) Medium preferences: $M = \{(l_i, l_j) \in \mathcal{L}_n^+ \times \mathcal{L}_n^+ \mid i + j = 3n + 1\}$.
(3) High preferences: $H = \{(l_i, l_j) \in \mathcal{L}_n^+ \times \mathcal{L}_n^+ \mid i + j > 3n + 1\}$.

Given a monotonic operator $F : \mathcal{L}_n^+ \times \mathcal{L}_n^+ \longrightarrow \mathcal{L}_n$, we can decompose F in three mappings, $F^L : L \longrightarrow \mathcal{L}_n$, $F^M : M \longrightarrow \mathcal{L}_n$ and $F^H : H \longrightarrow \mathcal{L}_n$, one for each kind of preferences in such a way that

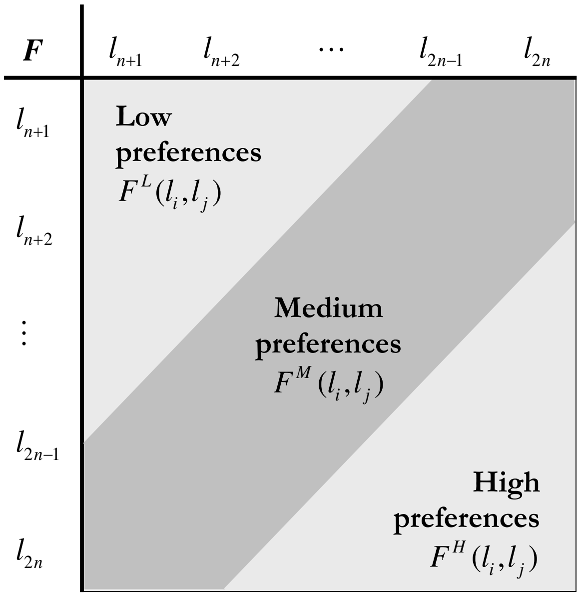
$$F(l_i, l_j) = \begin{cases} F^L(l_i, l_j), & \text{if } (l_i, l_j) \in L, \\ F^M(l_i, l_j), & \text{if } (l_i, l_j) \in M, \\ F^H(l_i, l_j), & \text{if } (l_i, l_j) \in H. \end{cases}$$

In Figure 2 we show the decomposition of F in F^L, F^M and F^H.

For $n = 4$, we have:

$$L = \{(l_5, l_5), (l_5, l_6), (l_5, l_7), (l_6, l_5), (l_6, l_6), (l_7, l_5)\}.$$

Fig. 2. Decomposition of F in F^L, F^M and F^H



$$M = \{(l_5, l_8), (l_6, l_7), (l_7, l_6), (l_8, l_5)\}.$$

$$H = \{(l_6, l_8), (l_7, l_7), (l_7, l_8), (l_8, l_6), (l_8, l_7), (l_8, l_8)\}.$$

In Table 6 we show the decomposition of F for $n = 4$.

Table 6
Decomposition of F for $n = 4$

F	l_5	l_6	l_7	l_8
l_5	$F^L(l_5, l_5)$	$F^L(l_5, l_6)$	$F^L(l_5, l_7)$	$F^M(l_5, l_8)$
l_6	$F^L(l_6, l_5)$	$F^L(l_6, l_6)$	$F^M(l_6, l_7)$	$F^H(l_6, l_8)$
l_7	$F^L(l_7, l_5)$	$F^M(l_7, l_6)$	$F^H(l_7, l_7)$	$F^H(l_7, l_8)$
l_8	$F^M(l_8, l_5)$	$F^H(l_8, l_6)$	$F^H(l_8, l_7)$	$F^H(l_8, l_8)$

In Table 7 we show the decomposition of F_k for $k = 1, 2, 3, 4$.

Table 7
Decomposition of F_k for $k = 1, 2, 3, 4$

k	$F_k^L(l_i, l_j)$	$F_k^M(l_i, l_j)$	$F_k^H(l_i, l_j)$
1	$\min\{l_i, l_j\}$	$\min\{l_i, l_j\}$	$\min\{l_i, l_j\}$
2	l_{n+1}	l_{n+1}	l_{i+j-2n}
3	l_n	l_{n+1}	l_{i+j-2n}
4	l_{i+j-2n}	l_{n+1}	l_{i+j-2n}

We now consider some monotonic operators which can be considered as appropriate for modeling rationality in the linguistic framework. The differences with respect to the monotonic operators included in 4.1 are double: weaker requirements in high preferences (H) and stronger requirements in low preferences (L). These changes guarantee P-transitivity in F_k^H for $k = 5, 6, 7, 8$ and in F_k^M for $k = 5, 6, 7$. Triple-acyclicity is guaranteed in F_6^L, but not in F_k^L for $k = 5, 7, 8$.

In Table 8 we show the decomposition of F_k for $k = 5, 6, 7, 8$, and Table 9 includes percentages of individuals satisfying F_k-transitivity for $k = 5, 6, 7, 8$.

Table 8
Decomposition of F_k for $k = 5, 6, 7, 8$

k	$F_k^L(l_i, l_j)$	$F_k^M(l_i, l_j)$	$F_k^H(l_i, l_j)$
5	l_{i+j-2n}	l_{n+1}	l_{n+2}
6	l_n	l_{n+1}	l_{n+2}
7	$\begin{cases} l_{n-1} \text{ if } \begin{cases} i+j=3n \text{ and} \\ i \neq 2n-1 \neq j \end{cases} \\ l_{i+j-2n}, \text{ otherwise} \end{cases}$	l_{n+1}	l_{n+2}
8	$\begin{cases} l_{n-1} \text{ if } \begin{cases} i+j=3n \text{ and} \\ i \neq 2n-1 \neq j \end{cases} \\ l_{i+j-2n}, \text{ otherwise} \end{cases}$	$\begin{cases} l_{n+1} \text{ if } i=2n \text{ or } j=2n \\ l_n, \text{ otherwise} \end{cases}$	l_{n+2}

Table 9
Percentages of individuals who fulfill F_k-transitivity for $k = 5, 6, 7, 8$

	F_5	F_6	F_7	F_8
X_1	84%	80%	84%	86%
X_2	92%	88%	94%	96%
X_3	90%	88%	92%	94%

We note that F_k-transitivity for $k = 4, 5$ do not ensure triple-acyclicity in the associated ordinary preferences. On the other hand, the only differences between F_4 and F_5 appear in high preferences: $F_5^H(l_i, l_j) = l_{n+2} \leq l_{i+j-2n} = F_4^H(l_i, l_j)$, for every $(l_i, l_j) \in H$. In this way, F_5-transitivity reaches the same or greater fulfillment percentages than F_4-transitivity (in X_1, and in X_2 and X_3, respectively).

The differences between F_5-transitivity and F_6-transitivity only appear in low preferences: $F_6^H(l_i, l_j) = l_n \geq l_{i+j-2n} = F_5^H(l_i, l_j)$, for every $(l_i, l_j) \in L$.

Nevertheless, F_6-transitivity reaches smaller fulfillment percentages than F_5-transitivity.

With respect to F_k-transitivity for $k = 7, 8$, we note that they do not guarantee triple-acyclicity in the associated ordinary preferences.

There exist small differences between F_7-transitivity and F_5-transitivity in low preferences: $F_7^L(l_i, l_j) = l_{n-1} < l_{i+j-2n} = F_5^L(l_i, l_j)$ whenever $i + j = 3n$ and $i, j \neq 2n - 1$.

Among the considered monotonic operators which ensure P-transitivity in medium and high preferences, F_7-transitivity reaches the highest fulfillment percentages.

F_8-transitivity is the only F_k-transitivity property which does not guarantee P-transitivity in the associated ordinary preferences for medium preferences. However, F_8-transitivity ensures P-transitivity in the associated ordinary preferences for high preferences. The differences between F_8-transitivity and F_7-transitivity only appear in medium preferences: $F_8^M(l_i, l_j) = l_n \leq l_{n+1} = F_7^M(l_i, l_j)$ whenever $i, j \neq 2n$. This smaller requirement entails the highest fulfillment percentages.

Remark 7 To analyze the statistical significance of the improvements from our models over those from previous models, we perform both parametric and non parametric tests to compare the proportion of successes obtained with models F_k $(k = 1, 2, ..., 8)$ to those obtained with complete preorders, that is regarded as our benchmark. In particular, we perform a parametric test to compare the population proportions from Bernoulli models based on the approximation to the Normal distribution, and also a sign test on the paired samples.

We have compared the fulfillment of $F_1, \ldots, F_8$ with the fulfillment of complete preorders. Our hypotheses have been H_0: $p_k^j \geq p_0^j$ versus H_1: $p_k^j < p_0^j$, being p_k^j =Prob(fulfillment of model F_k in set X_j), with $j = 1, 2, 3$ and $k = 1, 2, \ldots, 8$, and p_0^j =Prob(fulfillment of complete preorders in set X_j), with $j = 1, 2, 3$. The results from both tests have been very similar and the conclusions are clear: with a confidence level of 95%, the percentages of models F_k (except for F_1) are significantly higher than those obtained with the model based on complete preorders. In particular, the p-values of both tests for the models $F_5, \ldots, F_8$, in the three sets (X_1, X_2, X_3), are higher than 99, 9%. This fact proves, beyond any doubt, that $p_k^j \geq p_0^j$.

5 Concluding remarks

Since usually human being have difficulties for assigning exact numerical values to intensities of preference, linguistic preferences have the advantage of dealing with linguistic expressions such as occur in real life. In this paper we have considered that possibility and we have introduced a class of consistency models based on monotonic operators, which generalizes the class of t-norms in discrete settings. So, given a monotonic operator F on a subset of linguistic preferences, our model of coherence requires that if a is preferred to b and b is preferred to c, then the intensity of preference between a and c should be at least the outcome given by F to the preference intensities between a and b and b and c. Then, the problem is to determine a class of suitable monotonic operators for modeling rationality in the linguistic framework.

It is important to emphasize that our proposal takes into account real data provided by individuals in real decision problems. First of all, we have considered some classical model of rationality, and we can conclude that, according to the empirical data, neither linear orders nor complete preorders do not capture the real behavior of individuals. Moreover, the fulfillment of these rationality models decreases whenever individuals show low preferences among alternatives. Subsequently, we have introduced other models of rational behavior which take into account intensities of preference. In this sense, F-transitivity (based on monotonic operators) allows us to define a wide class of rationality models which differentiate low, medium and high preferences by considering different requirements in each kind of preference modalities. According to our empirical data, those F-transitivity properties which guarantee P-transitivity in medium and high preferences have obtained very good results.

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Appendix. Tables for X_1 and X_3

Table 10
Triple-acyclicity and P-transitivity in X_1

$R(a,b)$ $R(b,c)$	$R(a,c)$ l_0, l_1, l_2, l_3	$R(a,c)$ l_4	$R(a,c)$ l_5, l_6, l_7, l_8
l_5, l_5	2.53%	6.33%	91.14%
l_5, l_6	5.85%	5.26%	88.89%
l_5, l_7	3.80%	2.53%	93.67%
l_5, l_8	5.36%	5.36%	89.29%
l_6, l_6	1.39%	5.56%	93.06%
l_6, l_7	4.63%	1.85%	93.52%
l_6, l_8	4.84%	3.23%	91.94%
l_7, l_7	0.00%	3.13%	100%
l_7, l_8	0.00%	0.00%	100%
l_8, l_8	0.00%	0.00%	100%
Total	3.86%	4.01%	92.13%

Table 11
Triple-acyclicity and P-transitivity in X_3

$R(a,b)$ $R(b,c)$	$R(a,c)$ l_0, l_1, l_2, l_3	$R(a,c)$ l_4	$R(a,c)$ l_5, l_6, l_7, l_8
l_5, l_5	9.68%	6.45%	83.87%
l_5, l_6	1.54%	3.08%	95.38%
l_5, l_7	0.00%	1.30%	98.70%
l_5, l_8	0.00%	0.00%	100%
l_6, l_6	3.25%	0.00%	96.75%
l_6, l_7	1.14%	0.57%	98.30%
l_6, l_8	2.27%	0.00%	97.73%
l_7, l_7	1.85%	0.00%	98.15%
l_7, l_8	0.00%	0.00%	100%
l_8, l_8	0.00%	0.00%	100%
Total	1.86%	0.99%	97.15%