Defining the Borda count in a linguistic decision making context

J.L. García-Lapresta, M. Martínez-Panero, L.C. Meneses

Departamento de Economía Aplicada, Facultad de Ciencias Económicas y Empresariales, Universidad de Valladolid, Avda. Valle de Esgueva 6, 47011 Valladolid, Spain.

Abstract

Different kinds of decision rules have been successfully implemented under a linguistic approach. This paper aims the same goal for the Borda count, a well-known procedure with some interesting features. In order to this, two ways of extension from the Borda rule to a linguistic framework are proposed taking into account all the agents' opinions or only the favorable ones for each alternative when compared with each other. In the two cases, both individual and collective Borda counts are analyzed, asking for properties as good as those of the original patterns.

Key words: Linguistic labels, Linguistic preferences, Trapezoidal fuzzy numbers, Borda count, Condorcet winner, Condorcet loser.

1 Introduction

The Borda rule is an appropriate procedure in multi-person decision making when several alternatives are considered. This estimation relies on the processed information from the whole set of alternatives, not only from each agent's most preferred one (see Saari [29, p. 19] and Dummett [10], among others). In addition, Black [3,4], Mueller [27] and Straffin Jr. [31] have noted that the Borda count chooses the alternative which stands highest on average in the agents' preference orderings.

The literature about the Borda rule is very extensive (see Martínez-Panero [25] and Richter and Wong [28] for references). Of course, the original memory

Email addresses: `lapresta@eco.uva.es` (J.L. García-Lapresta), `panero@eco.uva.es` (M. Martínez-Panero), `lmeneses@eco.uva.es` (L.C. Meneses).

of Borda [5] must be mentioned, but a more comprehensive analysis, with adjustments in the case of indifference between alternatives, can be found in Black [3,4] and Gärdenfors [17], among others. However, this treatment is not faithful enough to the agents' opinions on the alternatives, due to the way of codifying their ordinary preferences, only through discrete values, namely 0, 0.5 and 1.

There exists a natural development which allows the agents to graduate their opinions on the alternatives by means of fuzzy preferences. This approach extends the above mentioned range to all possible values from 0 to 1, and provides a more versatile count than those based on ordinary preferences. On this gradual Borda rule and its variants, see Marchant [23,24], García-Lapresta and Martínez-Panero [15,16] and Martínez-Panero [25].

The inputs of the mentioned Borda counts are numerical assessments. However, the agents tend to operate (at least in a latent manner) in terms of linguistic expressions rather than with numbers (see Zadeh [35–37] and Herrera, Herrera-Viedma and Verdegay [18], among others). Well, this fact conducts us naturally to a linguistic framework. So, our aim in this paper consists in designing linguistic Borda counts from the original Borda rule and its discrete or gradual extensions, preserving their good features as far as possible (on this approach see García-Lapresta, Lazzari and Martínez-Panero [14]). Whichever the pattern may be (discrete or gradual), we note that the Borda rule is a two stage scheme in the following sense. In the first phase, individual Borda counts are computed, and it would be desirable for these scores to respect the agent's opinions on the alternatives. In García-Lapresta and Martínez-Panero [16] and Martínez-Panero [25] we have asked for the individual Borda counts to be representative of the corresponding preferences, and in connection with this aspect, different modalities of transitivity which ensure representativity for the individual counts have been analyzed, depending on the discrete or gradual version of the implemented procedure. The second phase aggregates individual scores into a total one for each alternative, and the highest score determines the Borda winner. We note that this scheme does not respect the Condorcet principle: the alternative which defeats each other by simple majority in pairwise tournaments might not be the Borda winner (in fact Gärdenfors [17] and Young [34] extend this result to any scoring rule, as Condorcet did). So, the Borda rule is not Condorcet consistent (see Baharad and Nitzan [1] for a deeper insight on this analysis). However, the Condorcet winner cannot be a Borda loser (see Fishburn and Gehrlein [12] for the original Borda rule, and Martínez-Panero [25] for the discrete and gradual cases).

The paper is organized as follows. In Section 2 we use linguistic labels for the agents to express their preferences. Then, we have represented the labels through trapezoidal fuzzy numbers, which capture the vagueness of such modality of preferences. The usual fuzzy arithmetic and a specific order are also established for computation. In Section 3, two kinds of both individual and collective linguistic Borda counts are introduced. Moreover, we find linguistic transitivity conditions which ensure these procedures be representative of the individual linguistic preferences. Subsection 3.2 is devoted to show, by means of an example, how to implement such linguistic Borda procedures. Subsection 3.3 includes the aforementioned Borda-Condorcet analysis. This is extended to a linguistic context, and a symmetry condition on the semantics associated with the set of labels is found to guarantee desirable properties for one of the introduced linguistic Borda counts. Finally, Section 4 contains some concluding remarks.

2 Linguistic preferences

Preferences are a basic instrument for dealing with decision problems. Although fuzzy preferences are a relevant tool for modelling preference intensities, linguistic preferences could be more appropriate for capturing the lack of precision in human behavior. Some papers related to the linguistic approach in decision making are Yager [33], Herrera, Herrera-Viedma and Verdegay [18,19], Herrera – Martínez [20], Herrera-Viedma [21], Herrera-Viedma and Peis [22], Batyrshin, Shajdullina and Shemeretov [2], Xu [32] and García-Lapresta [13], among others.

Now we take into account the approach included in García-Lapresta [13]. Let $X = \{x_1, \dots, x_n\}$ be a set of alternatives and assume that m agents show their preferences over the pairs of X in a linguistic manner, with $n \geq 2$ and $m \geq 2$. Let $L = \{l_0, l_1, \dots, l_s\}$ be a set of linguistic labels, where $s \geq 2$, ranked by a linear order: $l_0 < l_1 < \cdots < l_s$. There ought to be an intermediate label representing indifference, and the rest of labels are defined around it symmetrically. The number of labels, $s + 1$, will be odd and, consequently, $l_{s/2}$ is the central label.

Suppose that each agent $k \in \{1, \dots, m\}$ compares all the pairs of alternatives of X and declares levels of preference by means of a linguistic binary relation $R^k : X \times X \longrightarrow L$. In what follows we use the notation $r^k_{ij} = R^k(x_i, x_j)$ and it means the level of preference with which agent k prefers x_i over x_j.

Definition 1 *A* linguistic preference relation on X based on L *is a mapping* $R^k : X \times X \longrightarrow L$ *such that:*

$r^k_{ij} = l_s$, *if* x_i *is totally preferred to* x_j,

$l_{s/2} < r^k_{ij} < l_s$, *if* x_i *is somewhat preferred to* x_j,

$r_{ij}^k = l_{s/2}$, if x_i is indifferent to x_j,

$l_0 < r_{ij}^k < l_{s/2}$, if x_j is somewhat preferred to x_i,

$r_{ij}^k = l_0$, if x_j is totally preferred to x_i.

Remark 2 *On the sequel we will assume that linguistic preference relations satisfy the following* reciprocity *condition:*

$$r_{ij}^k = l_h \quad \Leftrightarrow \quad r_{ji}^k = l_{s-h},$$

for all $x_i, x_j \in X$ and all $h \in \{0, 1, \ldots, s\}$.

The Borda count requires these labels to be added, and the results to be compared. We note that linguistic labels can be managed symbolically by means of the linguistic OWA operators introduced in Herrera, Herrera-Viedma and Verdegay [19]. However, as pointed out before, in this paper we follow García-Lapresta [13]. In this way, we consider the commutative monoid $(\langle L \rangle, +)$ generated by L by means of all possible sums of labels of L with an associative and commutative operation $+$ on L, where l_0 is the neutral element:

(1) $L \subset \langle L \rangle$
(2) $l + l' \in \langle L \rangle$, for all $l, l' \in \langle L \rangle$
(3) $l + (l' + l'') = (l + l') + l''$, for all $l, l', l'' \in \langle L \rangle$
(4) $l + l' = l' + l$, for all $l, l' \in \langle L \rangle$
(5) $l + l_0 = l$, for all $l \in \langle L \rangle$.

We also consider a total order $\leq$ on $\langle L \rangle$ compatible with the original order on L:

(6) $l \leq l$, for all $l \in \langle L \rangle$
(7) $(l \leq l'$ and $l' \leq l) \Rightarrow l = l'$, for all $l, l' \in \langle L \rangle$
(8) $(l \leq l'$ and $l' \leq l'') \Rightarrow l \leq l''$, for all $l, l', l'' \in \langle L \rangle$
(9) $l \leq l'$ or $l' \leq l$, for all $l, l' \in \langle L \rangle$
(10) $l_0 < l_1 < \cdots < l_s$, where $<$ is the strict order associated with $\leq$ ($l < l'$ if $l \leq l'$ and $l \neq l'$, for all $l, l' \in \langle L \rangle$).

Moreover, we suppose

(11) $l \leq l' \Rightarrow l + l'' \leq l' + l''$, for all $l, l', l'' \in \langle L \rangle$.

Then, $(\langle L \rangle, +, \leq)$ is a totally ordered commutative monoid.

We note that for each $l \in \langle L \rangle$, there exist non negative integers $\lambda_0, \lambda_1, \ldots, \lambda_s$ such that $l = \sum_{k=0}^{s} \lambda_k l_k$, where $\lambda_k l_k$ is the addition of λ_k times the label l_k and $0 l_k = l_0$.

Although $(\langle L\rangle, +)$ is a canonical structure, $(\langle L\rangle, \leq)$ is not. So, we are going to represent the labels through trapezoidal fuzzy numbers (see, for instance, Dubois and Prade [9]), because this kind of fuzzy numbers are an appropriate tool to capture the vagueness of the assessments (see, for instance, Delgado, Vila and Voxman [6]).

Following the last mentioned authors, given a trapezoidal fuzzy number (TFN) $t = (a, b, c, d)$, its *value*, $V(t)$, and *ambiguity*, $A(t)$, are the real numbers

$$V(t) = \frac{c+b}{2} + \frac{(d-c)-(b-a)}{6}$$

and

$$A(t) = \frac{c-b}{2} + \frac{(d-c)+(b-a)}{6}.$$

For comparing TFNs, we consider a lexicographic order which attends to the highest value as first characteristic, and if the TFNs share the same value, the smallest ambiguity is taken into account, as corresponding to the sharpest aggregated opinion:

$$t \succ t' \quad \Leftrightarrow \quad \begin{cases} V(t) > V(t') \\ \text{or} \\ V(t) = V(t') \;\text{ and }\; A(t) < A(t'). \end{cases}$$

This binary relation between TFNs is asymmetric, i.e., $t \succ t'$ implies that not $t' \succ t$, and negatively transitive, i.e., if not $t \succ t'$ and not $t' \succ t''$, then not $t \succ t''$; consequently, $\succ$ is transitive. However, $\succ$ is not a linear order. For instance, neither $(0, 3, 3, 6) \succ (2, 2, 4, 4)$ nor $(2, 2, 4, 4) \succ (0, 3, 3, 6)$, since both TFNs have the same value, 3, and the same ambiguity, 1. If two different TFNs share value and ambiguity, it should be necessary to extend the lexicographic order above (see Delgado, Vila and Voxman [7]).

As mentioned before, we associate with each linguistic label $l_h \in L$ an appropriate TFN t_h. Furthermore, we extend the representation to $\langle L\rangle$: if $l = \sum_{h=0}^{s} \lambda_h l_h$, its associated TFN would be $t = \sum_{h=0}^{s} \lambda_h t_h$.

Since $V(t + t') = V(t) + V(t')$ and $A(t + t') = A(t) + A(t')$ for every pair of TFNs t and t', we can extend the above order $\succ$ in the set of TFNs to the set $\langle L\rangle$: $l \succ l' \Leftrightarrow t \succ t'$, where t and t' are the associated TFNs of l and l', respectively.

Definition 3 *Two TFNs* $t = (a, b, c, d)$ *and* $t' = (a', b', c', d')$ *are* symmetrical *if* $a' = 1-d$, $b' = 1-c$, $c' = 1-b$ *and* $d' = 1-a$. *Given a representation of*

$L = \{l_0, \ldots, l_s\}$ by means of TFNs $t_0, \ldots, t_s$, we say that this representation is symmetrical *if t_h and t_{s-h} are symmetrical, for each $h \in \{0, 1, \ldots, s\}$.*

It is worth to emphasize that for each symmetrical representation of L, it holds $t_{s/2} = (a, b, 1-b, 1-a)$ for some $a, b \in [0, 1]$, and $V(t_{s/2}) = 0.5$.

In Table 1 we introduce a concrete semantics which provides a symmetrical representation of a set with 9 linguistic labels. We note that ambiguity is greater around indifference than in the proximities of extreme preferences, which are crisp. This semantics will be used in the examples of the paper.

Label	**Meaning**	**TFN**
l_0	x_j is totally preferred to x_i	$t_0 = (0, 0, 0, 0)$
l_1	x_j is highly preferred to x_i	$t_1 = (0, 0.02, 0.05, 0.11)$
l_2	x_j is rather preferred to x_i	$t_2 = (0.05, 0.11, 0.17, 0.25)$
l_3	x_j is slightly preferred to x_i	$t_3 = (0.17, 0.25, 0.34, 0.44)$
l_4	x_i is indifferent to x_j	$t_4 = (0.34, 0.44, 0.56, 0.66)$
l_5	x_i is slightly preferred to x_j	$t_5 = (0.56, 0.66, 0.75, 0.83)$
l_6	x_i is rather preferred to x_j	$t_6 = (0.75, 0.83, 0.89, 0.95)$
l_7	x_i is highly preferred to x_j	$t_7 = (0.89, 0.95, 0.98, 1)$
l_8	x_i is totally preferred to x_j	$t_8 = (1, 1, 1, 1)$

Table 1: Semantics with 9 linguistic labels.

3 Linguistic Borda counts

Definition 4 *Let R^k be a linguistic preference relation on X based on L.*

(1) The ordinary preference relation associated with R^k, *$\succ_k$, is defined by $x_i \succ_k x_j$ if and only if $r_{ij}^k > l_{s/2}$.*

(2) The ordinary preference-indifference relation associated with R^k, *$\succeq_k$, is defined by $x_i \succeq_k x_j$ if and only if $r_{ij}^k \geq l_{s/2}$.*

The threshold $l_{s/2}$ is considered because the condition $r_{ij}^k > r_{ji}^k$ is equivalent to $r_{ij}^k > l_{s/2}$.

We note that $\succ_k$ is asymmetric, i.e., if $x_i \succ_k x_j$, then does not hold $x_j \succ_k x_i$; and $\succeq_k$ is complete, i.e, $x_i \succeq_k x_j$ or $x_j \succeq_k x_i$, for all $x_i, x_j \in X$.

3.1 Broad and narrow approaches

Now we introduce two different linguistic Borda counts, broad and narrow, both of them extending the original Borda count, depending on the aggregation of all the preference levels, or only those greater than the central label. It is worth mentioning that Sen [30] also defines broad and narrow Borda counts in connection with the fulfillment of the independence of irrelevant alternatives principle. However, these variants, although sharing names, are different of ours. Sen takes into account the amplitude of the referential set of alternatives in order to define a choice function, while we consider opinions (all of them or only the favorable ones) in pairwise comparisons of alternatives.

Definition 5 *Let $k \in \{1, \ldots, m\}$.*

(1) The broad Borda count of k *is the mapping $\overline{r}_k : X \longrightarrow \langle L \rangle$ defined by:*

$$\overline{r}_k(x_i) = \sum_{j=1}^{n} r_{ij}^k.$$

(2) The narrow Borda count of k *is the mapping $\widehat{r}_k : X \longrightarrow \langle L \rangle$ defined by:*

$$\widehat{r}_k(x_i) = \sum_{x_i \succ_k x_j} r_{ij}^k.$$

(3) The collective broad Borda count *is the mapping $\overline{\boldsymbol{r}} : X \longrightarrow \langle L \rangle$ defined by:*

$$\overline{\boldsymbol{r}}(x_i) = \sum_{k=1}^{m} \overline{r}_k(x_i).$$

(4) The collective narrow Borda count *is the mapping $\widehat{\boldsymbol{r}} : X \longrightarrow \langle L \rangle$ defined by:*

$$\widehat{\boldsymbol{r}}(x_i) = \sum_{k=1}^{m} \widehat{r}_k(x_i).$$

Definition 6 *Given $x_i \in X$,*

(1) x_i is a broad Borda winner *if $\overline{\boldsymbol{r}}(x_i) \geq \overline{\boldsymbol{r}}(x_j)$ for all $x_j \in X$.*

(2) x_i is a narrow Borda winner *if $\widehat{\boldsymbol{r}}(x_i) \geq \widehat{\boldsymbol{r}}(x_j)$ for all $x_j \in X$.*

(3) x_i is a broad Borda loser *if $\overline{\boldsymbol{r}}(x_i) \leq \overline{\boldsymbol{r}}(x_j)$ for all $x_j \in X$.*

(4) x_i is a narrow Borda loser *if $\widehat{\boldsymbol{r}}(x_i) \leq \widehat{\boldsymbol{r}}(x_j)$ for all $x_j \in X$.*

It would be desirable for the linguistic Borda counts to require the fulfillment of the following property of representativity: when two alternatives are compared by an agent, that with the highest Borda qualification must be the preferred one.

Definition 7 *Let R^k be a linguistic preference relation on X based on L.*

(1) The broad Borda count of the individual k is representative of R^k *if for all $x_i, x_j \in X$ it holds:*

$$x_i \succ_k x_j \;\Rightarrow\; \overline{r}_k(x_i) > \overline{r}_k(x_j).$$

(2) The narrow Borda count of the individual k is representative of R^k *if for all $x_i, x_j \in X$ it holds:*

$$x_i \succ_k x_j \;\Rightarrow\; \widehat{r}_k(x_i) > \widehat{r}_k(x_j).$$

Now we present individual rationality conditions in terms of some transitivity properties which ensure representativity of the corresponding Borda counts (on transitivity in the framework of linguistic preferences see, for instance, Díaz, García-Lapresta and Montes [8]).

Definition 8 *Let R^k be a linguistic preference relation on X based on L.*

(1) R^k is $\overline{T}$-transitive *if for all $x_i, x_j, x_p \in X$ it holds:*

$$(x_i \succeq_k x_j \textit{ and } x_j \succeq_k x_p) \;\Rightarrow\; r^k_{ip} \geq \max\{r^k_{ij}, r^k_{jp}\}.$$

(2) R^k is $\widehat{T}$-transitive *if for all $x_i, x_j, x_p \in X$ it holds:*

$$(x_i \succ_k x_j \textit{ and } x_j \succ_k x_p) \;\Rightarrow\; r^k_{ip} \geq \max\{r^k_{ij}, r^k_{jp}\}.$$

Notice that $\overline{T}$-transitivity implies $\widehat{T}$-transitivity. However, it can be shown that representativity of $\overline{r}_k$ does not imply that of $\widehat{r}_k$.

Proposition 9 *If R^k is $\overline{T}$-transitive, then $\overline{r}_k$ is representative of R^k.*

PROOF. Suppose $x_i \succ_k x_j$, i.e., $r^k_{ij} > l_{s/2}$. First let us see that $r^k_{ip} \geq r^k_{jp}$ holds for each $x_p \in X$. Several cases can be considered:

(1) If $r^k_{jp} \geq l_{s/2}$, then $r^k_{ip} \geq \max\{r^k_{ij}, r^k_{jp}\}$.

(2) If $r^k_{jp} < l_{s/2}$ and $r^k_{ip} \leq l_{s/2}$, then $r^k_{pi} \geq l_{s/2}$. Thus, $r^k_{pj} \geq \max\{r^k_{pi}, r^k_{ij}\}$, and again $r^k_{ip} \geq r^k_{jp}$.

(3) If $r^k_{jp} < l_{s/2}$ and $r^k_{ip} > l_{s/2}$, then $r^k_{ip} \geq r^k_{jp}$ holds trivially.

Even more, being $r_{ij}^k > l_{s/2}$, the inequality $r_{ip}^k \geq r_{jp}^k$ is strict at least for $p = i$ and $p = j$. Therefore,

$$\overline{r}_k(x_i) = \sum_{p=1}^{n} r_{ip}^k > \sum_{p=1}^{n} r_{jp}^k = \overline{r}_k(x_j).$$

Proposition 10 *If R^k is $\widehat{T}$-transitive, then $\widehat{r}_k$ is representative of R^k.*

PROOF. Let $P(i) = \{p \mid r_{ip}^k > l_{s/2}\}$ for each $x_i \in X$. Then,

$$\widehat{r}_k(x_i) = \sum_{p \in P(i)} r_{ip}^k \quad \text{and} \quad \widehat{r}_k(x_j) = \sum_{p \in P(j)} r_{jp}^k.$$

Suppose $r_{ij}^k > l_{s/2}$; now we prove $P(j) \subset P(i)$: if $p \in P(j)$, then $r_{jp}^k > l_{s/2}$; by hypothesis $r_{ij}^k > l_{s/2}$, hence $r_{ip}^k \geq \max\{r_{ij}^k, r_{jp}^k\}$, and then $p \in P(i)$. Notice that $P(j) \neq P(i)$: $r_{ij}^k > l_{s/2}$ implies $j \in P(i)$, and $r_{jj}^k = l_{s/2}$ implies $j \notin P(j)$. Consequently there are more terms in $\widehat{r}_k(x_i)$ than in $\widehat{r}_k(x_j)$. Moreover, each term in $\widehat{r}_k(x_i)$ is greater than or equal to the corresponding term in $\widehat{r}_k(x_j)$: if $p \in P(j) \subset P(i)$, then we have $r_{ip}^k \geq \max\{r_{ij}^k, r_{jp}^k\} \geq r_{jp}^k$ and, consequently,

$$\widehat{r}_k(x_i) = \sum_{p \in P(i)} r_{ip}^k > \sum_{p \in P(j)} r_{jp}^k = \widehat{r}_k(x_j).$$

3.2 A comparative example

Now we show how to implement the aforementioned treatments in order to obtain a collective decision. As will be shown, the final result may depend on the chosen procedure.

Example 11 Suppose three individuals who show their linguistic preferences over three alternatives, x_1, x_2 and x_3, by means of the linguistic labels provided in Table 1. As usual, their assessments can be arranged through matrices. In our case, let

$$\left(r_{ij}^1\right) = \begin{pmatrix} l_4 & l_4 & l_6 \\ l_4 & l_4 & l_6 \\ l_2 & l_2 & l_4 \end{pmatrix}, \quad \left(r_{ij}^2\right) = \begin{pmatrix} l_4 & l_6 & l_6 \\ l_2 & l_4 & l_4 \\ l_2 & l_4 & l_4 \end{pmatrix}, \quad \left(r_{ij}^3\right) = \begin{pmatrix} l_4 & l_2 & l_2 \\ l_6 & l_4 & l_5 \\ l_6 & l_3 & l_4 \end{pmatrix}$$

be such matrices of individual linguistic preferences (all of them $\overline{T}$-transitive).

For each alternative, the individual broad Borda counts can be obtained by adding up the entries of the corresponding row in the associated matrix of individual linguistic preference, and so collective broad Borda counts are the sum of the entries of the corresponding row along all these matrices.

Then, in our case the collective counts are:

$$\overline{\boldsymbol{r}}(x_1) = 2l_2 + 4l_4 + 3l_6,\ \overline{\boldsymbol{r}}(x_2) = l_2 + 5l_4 + l_5 + 2l_6,\ \overline{\boldsymbol{r}}(x_3) = 3l_2 + l_3 + 4l_4 + l_6.$$

Now, for comparing these results we use the TFN representation provided by Table 1. Some computations show that:

$$V(2t_2 + 4t_4 + 3t_6) = 4.856,\ V(t_2 + 5t_4 + t_5 + 2t_6) = 5.058,$$

$$V(3t_2 + t_3 + 4t_4 + t_6) = 3.585.$$

Consequently, $\overline{\boldsymbol{r}}(x_2) > \overline{\boldsymbol{r}}(x_1) > \overline{\boldsymbol{r}}(x_3)$, and x_2 is the broad Borda winner.

The individual narrow Borda counts can be obtained by adding up the entries greater than l_4 of the corresponding row in the associated matrix of individual linguistic preference, and so collective narrow Borda counts are the sum of the entries greater than l_4 of the corresponding row along all these matrices. The results would be:

$$\widehat{\boldsymbol{r}}(x_1) = 3l_6,\ \widehat{\boldsymbol{r}}(x_2) = l_5 + 2l_6,\ \widehat{\boldsymbol{r}}(x_3) = l_6.$$

In this case, without any consideration about value or ambiguity, it is easy to see (only with the original order of labels) that $\widehat{\boldsymbol{r}}(x_1) > \widehat{\boldsymbol{r}}(x_2) > \widehat{\boldsymbol{r}}(x_3)$. Thus, x_1 would be the narrow Borda winner.

3.3 Condorcet analysis

Simple majority is one of the voting systems more used in practice. According to this procedure, x_i defeats x_j if there are more voters who prefer x_i to x_j than those who prefer x_j to x_i. The concept of Condorcet winner is related to simple majority. A Condorcet winner is an alternative which defeats each other by simple majority. There is a widespread agreement in the fact that such alternative (supposed to exist) must be chosen in a collective decision (see Fishburn [11], for example). Analogously, a Condorcet loser is an alternative which is defeated by each other by simple majority.

Now we introduce two linguistic extensions of these concepts, corresponding to the broad and the narrow points of view, in order to contrast them with those concerning the linguistic Borda approach.

Definition 12

(1) The broad simple majority *is defined by the ordinary binary relation* $\overline{P_S}$ *given by:*

$$x_i \, \overline{P_S} \, x_j \quad \textit{if and only if} \quad \sum_{k=1}^{m} r_{ij}^k > \sum_{k=1}^{m} r_{ji}^k.$$

(2) The narrow simple majority *is defined by the ordinary binary relation* $\widehat{P_S}$ *given by:*

$$x_i \, \widehat{P_S} \, x_j \quad \textit{if and only if} \quad \sum_{x_i \succ_k x_j} r_{ij}^k > \sum_{x_j \succ_k x_i} r_{ji}^k.$$

Notice that both $\overline{P_S}$ and $\widehat{P_S}$ are asymmetric binary relations, but not necessarily acyclic.

Definition 13 *Given* $x_i \in X$*,*

(1) x_i *is a* broad Condorcet winner *if* $x_i \, \overline{P_S} \, x_j$ *for all* $x_j \in X \setminus \{x_i\}$.

(2) x_i *is a* narrow Condorcet winner *if* $x_i \, \widehat{P_S} \, x_j$ *for all* $x_j \in X \setminus \{x_i\}$.

(3) x_i *is a* broad Condorcet loser *if* $x_j \, \overline{P_S} \, x_i$ *for all* $x_j \in X \setminus \{x_i\}$.

(4) x_i *is a* narrow Condorcet loser *if* $x_j \, \widehat{P_S} \, x_i$ *for all* $x_j \in X \setminus \{x_i\}$.

We note that both the Condorcer winners and the Condorcer losers are unique, whenever they exist (the Condorcet approach might not be decisive in collective decision making).

Although Condorcet and Borda winners may be different, in the following proposition we justify that the broad Borda rule never selects the Condorcet loser when the set of labels is represented in a symmetrical way.

Proposition 14 *For each symmetrical representation of L, the broad Condorcet loser cannot be a broad Borda winner.*

PROOF. It is easy to check that for every couple of symmetrical TFNs $t = (a, b, c, d)$ and $t' = (1-d, 1-c, 1-b, 1-a)$, it holds $V(t) + V(t') = 1$ and $A(t) = A(t')$. Suppose that r_{ij}^k is represented by the TFN t_{ij}^k. We note that the total amount of the assessments made by an agent k is given by $\frac{n(n-1)}{2}$ couples $r_{ij}^k + r_{ji}^k$, where $i, j = 1, \ldots, n$ and $i \neq j$, plus n terms $r_{ii}^k = l_{s/2}$. Being 1 the value of each couple $t_{ij}^k + t_{ji}^k$ and 0.5 the value of the t_{ii}^k terms, the total value of the sum of all individual broad Borda counts of agent k is exactly $\frac{n(n-1)}{2} + n\,0.5 = \frac{n^2}{2}$. Then, the value of the sum of all collective broad Borda counts is $m\,\frac{n^2}{2}$. By way of contradiction, suppose x_i is both a broad Condorcet loser and a broad Borda winner. Bear in mind that $\sum_{k=1}^{m} r_{ij}^k$ and

$\sum_{k=1}^{m} r_{ji}^k$ are represented through sums of symmetrical TFNs, $\sum_{k=1}^{m} t_{ij}^k$ and $\sum_{k=1}^{m} t_{ji}^k$, respectively, with the same ambiguity. Consequently, if $\sum_{k=1}^{m} r_{ij}^k < \sum_{k=1}^{m} r_{ji}^k$, then $V\left(\sum_{k=1}^{m} t_{ij}^k\right) < V\left(\sum_{k=1}^{m} t_{ji}^k\right)$, if $i \neq j$. In addition, $V\left(\sum_{k=1}^{m} t_{ij}^k\right) + V\left(\sum_{k=1}^{m} t_{ji}^k\right) = m$, for all $j = 1, \ldots, n$. Then, the broad Condorcet loser must have a broad Borda count with a value smaller than $n\,\frac{m}{2}$. But, being x_i a broad Borda winner, none of the individual broad Borda counts of the n alternatives could reach the value $n\,\frac{m}{2}$; so the value of the sum of all collective broad Borda counts must be smaller than $n^2\,\frac{m}{2}$, contrary to the exact value found before.

Now we show that Proposition 14 is not verified under a narrow approach.

Example 15 Suppose five individuals who show their linguistic preferences over three alternatives x_1, x_2 and x_3, by means of the linguistic labels provided in Table 1, in the following manner: two of them prefer totally x_1 to both x_2 and x_3, and they have a slight preference for x_2 over x_3; the other three have a slight preference for x_2 over x_1 and for x_3 over x_1, and they are indifferent between x_2 and x_3. We note that all the individual preferences are $\widehat{T}$-transitive. It is easy to check that x_1 is simultaneously the narrow Condorcet loser and the narrow Borda winner.

By using similar arguments to those in the proof of Proposition 14, we obtain a dual result: the broad Borda count never ranks the broad Condorcet winner in the last position.

Proposition 16 *For each symmetrical representation of L, the broad Condorcet winner cannot be a broad Borda loser.*

Now we justify that Proposition 16 is not verified in the narrow cases.

Example 17 Suppose ten individuals who show their linguistic preferences over three alternatives x_1, x_2 and x_3, by means of the linguistic labels provided in Table 1, in the following manner: two of them highly prefer x_1 to both x_2 and x_3, and they have a slight preference for x_2 over x_3; three of them have a high preference for x_2 over x_1 and for x_3 over x_1, and they have a slight preference for x_3 over x_2; other three have a high preference for x_1 over x_3 and for x_2 over x_3, and they have a slight preference for x_1 over x_2; the last two individuals highly prefer x_3 to both x_1 and x_2, and they have a slight preference for x_2 over x_1. We note that all the individual preferences are $\widehat{T}$-transitive. It is easy to see that x_2 is simultaneously the narrow Condorcet winner and the narrow Borda loser.

4 Concluding remarks

The discrete Borda count allows the agents only to show which alternatives are preferred in pairwise comparisons. Thus, the use of a fuzzy approach does improve the method by considering more accurate information from the agents, namely, intensities of preference among alternatives. These assessments are usually required to be allocated in the unit interval. However, as Zadeh [36] suggests, it would be desirable to manage primary perceptions rather than measurements based on them. So, we use linguistic labels as inputs in our Borda count proposals. Two ways of implementation of a linguistic Borda count, broad and narrow, have been developed, taking into account all the assessments or only those favorable for one alternative when compared with each other. Concerning to the individual counts, the broad approach requires a more restrictive condition of linguistic transitivity than that founded for the narrow case in order to guarantee the representativity of the individual preferences. From a collective point of view, our Condorcet analysis has revealed that the linguistic broad Borda count presents better properties than the narrow one.

To sum up, the linguistic treatment appears to be close to the manner in which agents feel their preferences and, when applied to the Borda count, interesting features from the discrete or gradual cases, such as decisiveness, respect for the mean, etc., can be preserved. These are the reasons why we suggest the use of the linguistic Borda count in practice. And more specifically we advocate for the broad approach, which excludes Condorcet losers as the discrete and fuzzy Borda count do.

Acknowledgements

The financial support of the Spanish Ministry of Education and Science (Project SEJ2005-04392/ECON), the Junta de Castilla y León (Consejería de Educación y Cultura, Project VA092A08), and ERDF are gratefully acknowledged. The authors are also grateful for comments from Enrique Herrera-Viedma and two anonymous referees.

References

[1] E. Baharad, S. Nitzan (2003). The Borda rule, Condorcet consistency and Condorcet stability. *Economic Theory*, 22, pp. 685-688.

[2] I. Batyrshin, N. Shajdullina, L. Shemeretov (2004). Strict valued preference relations and choice function in decision-making procedures. In: R. Monroy, G. Arroyo-Figueroa, L.E. Sucar *et al.* (Eds.), *Lecture Notes in Computer Science*, 2972, Springer Verlag, Heidelberg, pp. 332-341.

[3] D. Black (1958). *The Theory of Committees and Elections.* Kluwer Academic Publishers, Boston.

[4] D. Black (1976). Partial justification of the Borda count. *Public Choice*, 28, pp. 1-16.

[5] J.C. de Borda (1784). *Mémoire sur les élections au scrutin.* Historie de l'Academie Royale des Sciences, Paris. Avaliable in McLean – Urken [26, pp. 81-89].

[6] M. Delgado, M.A. Vila, W. Voxman (1998). On a canonical representation of fuzzy numbers. *Fuzzy Sets and Systems*, 93, pp. 125-135.

[7] M. Delgado, M.A. Vila, W. Voxman (1998). A fuzzyness measure for fuzzy numbers: Applications. *Fuzzy Sets and Systems*, 94, pp. 205-216.

[8] S. Díaz, J.L. García-Lapresta, S. Montes (2008). Consistent models of transitivity for reciprocal preferences on a finite ordinal scale. *Information Sciences*, 178, pp. 2832-2848.

[9] D. Dubois, H. Prade (1987). Fuzzy numbers: An overview. In: J.C. Bezdek (Ed.), *Analysis of Fuzzy Information*, vol. 1. CRC Press, Boca Raton, pp. 3-39.

[10] M. Dummett (1998). The Borda count and agenda manipulation. *Social Choice and Welfare*, 15, pp. 289-296.

[11] P.C. Fishburn (1977). Condorcet social choice functions. *SIAM Journal on Applied Mathematics*, 33, pp. 469-489.

[12] P.C. Fishburn, W.V. Gehrlein (1976). Borda's rule, positional voting and Condorcet's simple majority principle. *Public Choice*, 28, pp. 79-88.

[13] J.L. García-Lapresta (2006). A general class of simple majority decision rules based on linguistic opinions. *Information Sciences*, 176, pp. 352-365.

[14] J.L. García-Lapresta, L.L. Lazzari, M. Martínez-Panero (2001). A group decision making method using fuzzy triangular numbers. In: C. Zopounidis, P.M. Pardalos and G. Baourakis (Eds.), *Fuzzy Sets in Management, Economics and Marketing.* World Scientific, Singapore, pp. 35-50.

[15] J.L. García-Lapresta, M. Martínez-Panero (2002). Borda count versus approval voting: A fuzzy approach. *Public Choice*, 112, pp. 167-184.

[16] J.L. García-Lapresta, M. Martínez-Panero (2002). A fuzzy Borda count in multi-person decisin making. In: T. Trzaskalik and J. Michnik (Eds.), *Multiple Objective and Goal Programming. Recent Developments.* Advances in Soft Computing, Springer Verlag, Berlin, pp. 46-60.

[17] P. Gärdenfors (1973). Positionalist voting functions. *Theory and Decision*, 4, pp. 1-24.

[18] F. Herrera, E. Herrera-Viedma, J.L. Verdegay (1995). A sequential selection process in group decision making with a linguistic assessment approach. *Information Sciences*, 85, pp. 223-239.

[19] F. Herrera, E. Herrera-Viedma, J.L. Verdegay (1996). Direct approach processes in group decision making using linguistic OWA operators. *Fuzzy Sets and Systems*, 79, pp. 175-190.

[20] F. Herrera, L. Martínez (2000). A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 8, pp. 746-752.

[21] E. Herrera-Viedma (2001). Modeling the retrieval process for an information retrieval system using an ordinal fuzzy linguistic approach. *Journal of the American Society for Information Sciences and Technology*, 52, pp. 460-475.

[22] E. Herrera-Viedma, E. Peis (2003). Evaluating the informative quality of documents in SGML format from judgements by means of fuzzy linguistic techniques based on computing with words. *Information Processing and Management*, 39, pp. 233-249.

[23] T. Marchant (1996). Valued relations aggregation with the Borda method. *Journal of Multi-Criteria Decision Analysis*, 5, pp. 127-132.

[24] T. Marchant (2000). Does the Borda rule provide more than a ranking? *Social Choice and Welfare*, 17, pp. 381-391.

[25] M. Martínez-Panero (2004). *Generalizaciones y Extensiones de la Regla de Votación de Borda*. Ph.D. Thesis, Universidad de Valladolid.

[26] I. McLean, A.B. Urken (eds.) (1995). *Classics of Social Choice*. The University of Michigan Press, Ann Arbor.

[27] D.C. Mueller (1979). *Public Choice*. Cambridge University Press, London.

[28] M.K. Richter, K.C. Wong (2008). Preference densities and social choices. *Economic Theory*, 36, pp. 225-238

[29] D.G. Saari (1995). *Basic Geometry of Voting*. Springer - Verlag, Berlin.

[30] A.K. Sen (1977). Social Choice Theory: A re-examination. *Econometrica*, 45, pp. 53-89.

[31] P.D. Straffin Jr. (1980). *Topics in the Theory of Voting*. Birkhäuser, Boston.

[32] Z. Xu (2004). A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Information Sciences*, 166, pp. 19-30.

[33] R.R. Yager (1995). An approach to ordinal decision making. *International Journal of Approximate Reasoning*, 12, pp. 237-261.

[34] H.P. Young (1995). Optimal voting rules. *Journal of Economic Perspectives*, 9, pp. 51-64.

[35] L.A. Zadeh (1975). The concept of a linguistic variable and its applications to approximate reasoning. *Information Sciences*. Part I: 8, pp. 199-249. Part II: 8, pp. 301-357. Part III: 9, pp. 43-80.

[36] L.A. Zadeh (2001). From computing with numbers to computing with words - From manipulation of measurements to manipulation of perceptions. In: P.P. Wang (Ed.), *Computing with Words*. John Wiley & Sons, New York, pp. 35-68.

[37] L.A. Zadeh (2008). Is there a need for fuzzy logic? *Information Sciences*, 178, pp. 2751-2779.