Satisfaction Approval Voting

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Overview

Satisfaction approval voting (SAV) is a voting system applicable to *multiwinner* elections (e.g., to a council or legislature). It uses an *approval ballot*, whereby voters can approve of as many candidates as they like (no rankings).

A voter's *satisfaction score* is the fraction of his or her approved candidates who are elected. If *k* candidates are to be elected, SAV chooses the set of *k* candidates that maximizes the *sum* of all voters' satisfaction scores.

Advantages:

• SAV is independent of the number of candidates a voter approves of—it works equally well for voters who are discriminating and not-so-discriminating in their choices.

• It tends to elect a more "representative" set of candidates than approval voting (AV)—in fact, SAV and AV may elect disjoint subsets—and inhibits clones.

• It can be applied to party-list systems, wherein it gives parties approximate proportional representation (PR).

• Because SAV favors larger parties, it gives parties an incentive to share support, form alliances, or even merge, perhaps into as few as two broad coalitions.

• Because voters can approve of multiple parties, parties are likely to be responsive to voter preferences in forming coalitions.

Other Methods for Aggregating Approval Votes

Brams (2008, chs. 4, 5, and 6) summarized and compared three different procedures for aggregating approval votes in multiwinner elections:

1. **Constrained AV** (Brams, 1990). Candidates are classified into categories (e.g., by party and by region), and their approval votes are aggregated by (i) electing candidates who receive the most votes in each sufficiently large category; and then (ii) electing candidates who receive the most votes overall. For example, half the winners might be elected according to (i) and half according to (ii), which is determined endogenously.

2. **Minimax procedure** (Brams, Kilgour, and Sanver, 2007). There are no categories, but the set of candidates that is elected is chosen to minimize the maximum weighted Hamming distance, where the weights reflect the connectedness of each voter to every other voter. This procedure ensures that no voter’s ballot, especially if he or she is well connected, is “too far” away from the outcome. Thus, the outcome is a compromise that reflects different interests, weighted by their centrality, in the electorate.

3. **Procedures that minimize misrepresentation** (Potthoff and Brams, 1998). Integer programs are used to determine the set of candidates that minimize misrepresentation, according to different measures of misrepresentation. These are based on approval votes or some other scoring system (e.g., the Borda count).

Additional procedures to discount approval votes that impair the ability of parties to elect more and more candidates—which are related to "size approval voting" (Alcalde-Unzu and Vorsaz, 2009)—include the following:

4. Proportional AV (http://www.nationmaster.com/encyclopedia/Proportional-approval-voting). Proposed by mathematician Forest Simmons in 2001, it considers all possible outcomes in which one or more candidates are elected. Each voter is given a "satisfaction score" for each outcome—based on how many candidates he or she approves of—getting 1 point for having one candidate elected, an extra 1/2 point for having a second candidate elected, and so on according to the following formula: 1 + 1/2 + 1/3 + ... The satisfaction of all voters when the required number of candidates is elected is summed, and the outcome with the highest satisfaction score is selected.

5. Sequential Proportional AV (http://www.nationmaster.com/encyclopedia/Sequential-proportional-approval-voting). Proposed by Danish statistician Thorvald N. Thiele in the early 1900s, the first candidate elected is the AV winner (w_1). The value of all ballots that approve of w_1 are reduced in value from 1 to 1/2 and the approval scores recalculated. Next, the unelected candidate who has the highest approval score is elected (w_2). Then the value of ballots that approve of w_1 and w_2 are reduced in value to 1/3, and the value of all ballots that approve of either w_1 or w_2 but not both are reduced in value to 1/2.

At each stage, the unelected candidate with the highest approval score is elected. Then the value of each voter's ballot is set at $1/(1+m)$—where m is the number of candidates approved on that ballot who were already elected—until the required number of candidates is elected.

Comparisons

None of these procedures is ideal for electing candidates in a parliamentary system if the goal is to ensure the proportional representation (PR) of parties:

• Constrained AV (1) fails, because PR is based on categories—and only partially at that—about which there may be little or no agreement as to which are most salient.

• The minimax procedure (2), while well suited to elect a set of *individuals* who are representative, provides no guarantee that *parties* will be proportionally represented.

• While procedures that minimize misrepresentation (3) do offer this guarantee, it depends on a particular measure of misrepresentation, about which there may be disagreement.

• While proportional AV (4) and sequential proportional AV (5) approximate PR, there is no proof that the apportionments they give satisfy properties considered desirable in an apportionment method (Balinski and Young, 1982/2001). These systems are also vulnerable to "favorite betrayal" (Brams, 2009)—whereby approving of a favorite candidate can hurt that candidate—and encourages clones.

SAV: Voting for Individual Candidates

Proposition 1. *If there are one or more candidates to be elected using approval balloting, SAV and AV may produce different winners.*

Example 1a: 9 voters, 3 candidates, *k* = 1 winner

5 voters: *ab*
4 voters: *c*

AV outcome: $\{a\}$ or $\{b\}$ (**5** votes each)
SAV outcome: $\{c\}$, because its satisfaction score (s) is greater than that of $\{a\}$ or $\{b\}$:

$s(c) = 5(0) + 4(1) = \mathbf{4}$
$s(a) = s(b) = 5(1/2) + 4(0) = \mathbf{2\frac{1}{2}}$.

Whereas *a* or *b* gives 5 voters partial satisfaction of ½, *c* gives 4 voters full satisfaction of 1.

Example 1b: 9 voters, 3 candidates, *k* = 2 winners

$s(a, c) = s(b, c) = 5(\frac{1}{2}) + 4(1) = \mathbf{6\frac{1}{2}}$
$s(a, b) = 5(1) + 4(0) = \mathbf{5}$.

Because $\{a, c\}$ or $\{b, c\}$ either partially or fully satisfy *all* 9 voters—whereas $\{a, b\}$ fully satisfies only a bare majority of 5 voters—the former would seem to be the more "representative" outcome, as well the one that maximizes total voter satisfaction.

Proposition 2. *Under SAV, the k winners are the k candidates who, individually, have the highest satisfaction scores*.

In Example 1, $s(a)$ = **2½**, $s(b)$ = **2½,** and $s(c)$ = **4**. Thus, when $k = 1$, $\{c\}$ is the winner, and when $k = 2$, $\{a, c\}$ or $\{b, c\}$ is the winner.

The *additivity* of candidate satisfaction scores follows from the fact that voter satisfaction is rooted in a relative rather than an absolute measure. The contribution of each voter to a candidate's score is $1/n$, where n is the number of candidates approved of by the voter.

If the contributions varied—as they do under proportional AV and sequential proportional AV—according to the number of *other* candidates in the winning set that a voter approves of, the satisfaction scores would *not* be additive.

Proposition 3. *There are outcomes in which AV and SAV elect disjoint subsets of candidates*.

Example 2: 10 voters, 4 candidates, *k* = 2 winners

4 voters: *ab*
3 voters: *c*
3 voters: *d*

AV outcome: $\{a, b\}$ (**4** votes each)
SAV outcome: $\{c, d\}$ [3(1) = **3** > 4(½) = **2**]

The *representativeness of an outcome* is the number of voters who approve of at least one winner.

Proposition 4. *There are examples in which neither SAV nor AV gives the most representative outcome.*

Example 3: 12 voters, 5 candidates, *k* = 2 winners

4 voters: *ab*
4 voters: *acd*
3 voters: *ade*
1 voter: *e*

AV outcome: {*a*, *d*} (**11** and **7** votes, respectively)
SAV outcome: {*a*, *d*}, because

$s(a) = 4(½) + 7(1/3) =$ **4 1/3**
$s(b) = 4(½) = 2$
$s(c) = 4(1/3) = 1\ 1/3$
$s(d) = 7(1/3) =$ **2 1/3**
$s(e) = 3(1/3) + 1(1) = 2$

But {*a*, *d*} is *not* the most representative outcome, because the voter who bullets votes for *e* receives no satisfaction from {*a*, *d*}.

By contrast {*a*, *e*} gives *some* representation to all 12 voters—each approves at least one of its candidates—though it does not maximize total voter satisfaction.

SAV *usually* represents at least as many, and often more, voters than AV, making it more representative.

This is so because the candidates that tend to benefit under SAV often have distinctive appeals that attract bullet voters. But these voters may not be numerous enough to outscore the voters that approve of several mainstream candidates and would win under AV.

However, there are (perhaps unlikely) exceptions:

Proposition 5. *There are examples in which AV gives a more representative outcome than SAV.*

Example 4: 13 voters, 5 candidates, k = 2 winners

2 voters: *a*
5 voters: *ab*
6 voters: *cde*

AV outcome: $\{a, c\}$, $\{a, d\}$, or $\{a, e\}$ (**7** votes for *a*, **6** votes for each of *c*, *d*, and *e*)
SAV outcome: $\{a, b\}$, because

$s(a) = 2(1) + 5(½) = \mathbf{4½}$
$s(b) = 5(½) = \mathbf{2½}$
$s(c) = s(d) = s(e) = 1\ 1/3$

Whereas AV gives some representation to all 13 voters, SAV gives representation to only 7 voters, because *b*'s satisfaction contribution is greater than *c*'s, *d*'s, or *e*'s.

This "problem" for SAV in Example 4 would disappear if *c*, *d*, and *e* combined forces and became one candidates (say, *c*), rendering $s(c) = 6(1) = 6$.

Then the SAV (and AV) outcomes would be $\{a, c\}$, which would give representation to all 13 voters.

Clones

AV, proportional AV, and sequential proportional AV create an incentive for clones to form, whereas SAV inhibits clones.

Recall Example 1b with $k = 2$ winners:

5 voters: *ab*
4 voters: *c*

The SAV outcome is $\{a, c\}$ or $\{b, c\}$. But if *c* becomes two clones, c_1 and c_2, then the outcome is $\{a, b\}$, because

$$s(a) = s(b) = 5(½) = \mathbf{2½} > s(c_1) = s(c_2) = 4(½) = \mathbf{2}.$$

Thus, *c* is *hurt* under SAV by creating c_1 and c_2, going from one winner, *c*, without a clone to none with a clone.

For *c* to benefit under SAV from clones, there would need to be at least 6 instead of 4 *c* supporters, because then $s(c_1) = s(c_2) = 6(½) = \mathbf{3}$, making the outcome $\{c_1, c_2\}$. While *c* "wastes" votes by running alone in this case, how many candidates—if *c* were a party—should it run?

SAV: Voting for Political Parties

In most party-list systems, voters vote for political parties, which win seats in a parliament in proportion to the number of votes they receive.

Under SAV, voters would *not* be restricted to voting for one party but could vote for as many parties as they like. If a voter approves of n parties, each approved party receives $1/n$ of a vote from that voter.

Unlike standard apportionment methods, SAV does *not* award seats according to the "quota" to which each party is entitled (typically, a whole number and a fractional remainder). Instead, parties receive seats that maximize total voter satisfaction, based on the fraction of their *nominated* candidates who are elected.

The number of candidates that a party nominates is equal to its quota *rounded up*.

Example 5: 13 voters, 3 parties, k = 3 winners

2 voters: *A*
5 voters: *B*
6 voters: *C*

Bullet Voting

Assume the supporters of each party bullet vote for it. Party *i*'s *quota*, q_i, is its proportion of votes (of the 11) times the number of seats to be apportioned:

$q_A = (5/11)(3) \approx 1.364$
$q_B = (4/11)(3) \approx 1.091$
$q_C = (2/11)(3) \approx 0.545.$

Rounded up to the next integer, *A*, *B*, and *C* nominate 2, 2, and 1 candidates, respectively.

The number of nominees is always greater than the number of candidates who can be elected, except when there is no rounding because the quotas are integers.

In Example 5, the number of nominees sum to 2 + 2 + 1 = 5, which is 2 greater than the number to be elected (also characteristic of cumulative voting).

SAV calculates satisfaction scores for all *plausible* apportionments of seats, which satisfy the following rules:

- No party receives more seats than its quota rounded up.
- The total for all parties sums to the number of seats to be apportioned.
- The largest party receives at least as many seats as the second-largest, the second-largest at least as many seats as third-largest, and so on.
- If there are ties, a tie-breaking procedure is used to determine which of the tied parties receives an extra seat if not all the tied parties can obtain one.

In Example 5, these rules yield the following satisfaction scores for the two plausible apportionments to parties (*A*, *B*, *C*):

$s(2, 1, 0) = 5(1) + 4(½) + 2(0) = 7$
$s(1, 1, 1) = 5(½) + 4(½) + 2(1) = 6½.$

In the case of apportionment (2, 1, 0), for example, the 5 *A* voters receive satisfaction of 1 for getting *A*'s 2 nominees elected, the 4 B voters satisfaction of ½ for getting 1 of *B*'s 2 nominees elected, and the 1 *C* voter satisfaction of 0 for not getting its single nominee elected.

Recall that when electing individual candidates, SAV chooses those with the greatest satisfaction scores. By comparison, when electing parties, SAV finds *apportionments* of seats to parties that maximize total voter satisfaction.

The SAV apportionment of seats to (*A*, *B*, *C*) also *satisfies quota*, because each party gets its exact quota, rounded either up or down.

But this apportionment is not an apportionment according to the *Hamilton method* (also called "largest remainders"), which begins by giving each party the integer portion of its exact quota (1 seat to *A* and 1 seat to *B*).

Any remaining seats to be apportioned go to the parties with the largest remainders until the seats are exhausted.

This means that that *C*, with the largest remainder (.545), gets the 3rd seat, yielding the apportionment (1, 1, 1). Hence, SAV does not give a Hamilton apportionment in this example.

Besides Hamilton, there are five so-called divisor methods of apportionment (Balinski and Young, 1982/2001). Among these, only the *Jefferson/d'Hondt* method—which is the one that most favors large parties—gives the SAV apportionment of (2, 1, 0) in Example 5.

Like Jefferson/d'Hondt, SAV also favors large parties. Unlike Jefferson/d'Hondt, it satisfies *upper quota*, because parties cannot nominate, and therefore cannot receive, more seats than their quotas rounded up.

An apportionment is *individually stable* if the supporters of any party, whom we assume act together, cannot increase their apportionment by approving of another party.

This is true of a SAV apportionment: If a party approves of another party, it increases the satisfaction score of that party and decreases its own score (see below)

C Supporters Also Approve of *B*

Assume that the 2 supporters of C—which received no seats when its supporters bullet vote for *C*—also approve of *B*. Then they increase B's satisfaction score by 2(½) **=** 1 and decrease C's satisfaction score also by 1, yielding

$q_A = (5/11)(3) \approx 1.364$
$q_B = (5/11)(3) \approx 1.364$
$q_C = (1/11)(3) \approx 0.273.$

The new quotas for *B* and *C* now allow an additional apportionment, (1, 2, 0), because of the tie in scores between *A* and *B*.

The resulting satisfaction scores for the three plausible apportionments are as follows:

$s(2, 1, 0) = 5(1) + 6(½) + 2(½) = \mathbf{9}$
$s(1, 2, 0) = 5(1) + 6(½) + 2(½) = \mathbf{9}$
$s(1, 1, 1) = 5(½) + 6(½) + 2(1) = \mathbf{7½.}$

The fact that both (2, 1, 0) and (1, 2, 0) maximize total voter satisfaction illustrates that *C* supporters—without changing *C*'s apportionment of 0—may change the apportionment of *A* and *B*, because now *B* instead of *C* may get 2 seats.

Proposition 6. *When the supporters of one party approve of another, they cannot increase their own party's apportionment but may affect the apportionments of other parties.*

B and *C* Supporters Approve of Each Other's Party

Now assume that *B* and *C* reach an agreement on policy issues, so their supporters now approve of each other's party. Because each party receives a satisfaction score of 6(½) = 3, their quotas are the following:

$q_A = (5/11)(3) \approx 1.364$
$q_B = (3/11)(3) \approx 0.818$
$q_C = (3/11)(3) \approx 0.818.$

These quotas allow for three plausible apportionments:

$s(2, 1, 0) = 5(1) + 4(½) + 2(½) = \mathbf{8}$
$s(2, 0, 1) = 5(1) + 4(½) + 2(½) = \mathbf{8}$
$s(1, 1, 1) = 5(½) + 4(1) + 2(1) = \mathbf{8½.}$

Thus, the SAV outcome that maximizes voter satisfaction is (1, 1, 1), so *A* loses a seat, *B* stays the same as when it bullet voted, and *C* gains a seat.

Together, *B* and *C* ensure themselves of a majority of seats that *A* previously obtained when *B* and *C* bullet voted, proving the following:

Proposition 7. *In a three-party system, if the supporters of the two smaller parties approve of each other's party, they may win a majority of seats when otherwise the large party would win a majority*.

A Paradox

Proposition 7 notwithstanding, the supporters of *B* and *C* may *not* approve of each other's party, because they do not *individually* benefit from doing so (only *collectively*).

Therefore, despite the fact that *B* and *C* supporters can together ensure themselves of 2 seats if they approve of each other's party, such "cooperative" strategies may not be chosen.

A possible way around this paradox is that *B* and *C* become one party—at least if they are ideologically compatible—reducing the party system to just two parties.

Because the combination of *B* and *C* has more supporters that *A* does, this combined party would win a majority of seats.

Conclusions

1. SAV is applicable to multiwinner elections. It uses an approval ballot—whereby voters can approve of as many candidates or parties as they like—but they are not given seats based on the number of approval votes they receive.

2. SAV measures the satisfaction of a *voter* by the fraction of his or her approved candidates that are elected. The set of candidates that maximizes the sum of voter satisfaction scores is selected.

3. This measure is independent of the number of candidates a voter approves of—it works equally well for voters who approve of few or many candidates—and so, in a sense, mirrors a voter's *personal* tastes. SAV may elect a completely different set of candidates from AV.

4. The satisfaction score of a *candidate* is the sum of the satisfaction contributions he or she receives from all voters. This is 1/*n* from each voter who approves of him or her, where *n* is the number of candidates approved of by the voter.

5. These equal contributions of voters to candidates make the winning set of candidates those with the highest individual satisfaction scores. This renders SAV outcomes easy to compute.

6. SAV tends to elect candidates that give more voters either partial or complete satisfaction—and thus representation—than does AV, but this is not universally true and is a question that deserves to be investigated further.

8. It would be interesting to apply SAV to the 2003 Game Theory Society (GTS) election, in which—using AV—161 voters elected 12 of 24 candidates to the GTS Council. Under the minimax procedure, 4 of the 12 winners would have been displaced by less popular candidates, including the candidate who received the fewest approval votes!

9. SAV inhibits candidates from creating clones to increase their representation, which they would have an incentive do under AV, proportional AV, and sequential proportional AV.

10. Because bullet voting is risky when voting for individual candidates (a voter's satisfaction score will be either 0 or 1), a risk-averse voter may be inclined to approve of multiple candidates.

11. When SAV is applied to party-list systems, the satisfaction score of a voter is the fraction of each party's *nominated* candidates he or she approves of who are elected. Each party is assumed to nominate a number of candidates equal to its quota rounded up.

12. The apportionment of seats to parties is never greater than a party's quota rounded up, but it need not be a Hamilton apportionment. Like the Jefferson/d'Hondt divisor method, SAV tends to favor larger parties.

13. Individually, parties are hurt when their supporters approve of other parties. Collectively, however, they may be able to increase their combined seat share by forming coalitions—whose supporters approve of all parties in it—or even merging.

14. Some voters may prefer to vote for single parties, whereas others may prefer to approve of multiple parties that, ultimately, may form a coalition government.

15. Voters might also be given the opportunity to use SAV not only to vote for parties but also to vote for the *nominees* of a favorite party.

16. Parties can benefit from coordinating their policies and forming coalitions to increase their combined seat share, possibly reducing the party system two broad left-of-center and right-of-center parties or coalitions of parties.

17. Alternatively, a third moderate party might emerge (e.g., Kadima in Israel) that peels away supporters from the left and the right. This seems all very democratic, making coalitions fluid and responsive to popular sentiment.

18. More coordination by the parties would give voters a better idea of what to expect when they decide which parties to support, compared with the situation today when voters can never be sure about what parties will join in a governing coalition and what its policies will be.

19. Because SAV makes it easier for voters to know what parties to approve of, and for party coalitions to form that reflect their interests, SAV should lead to more informed voting and more responsive government.