SSEAC Workshop

Alternatives to Ranking in the Theory of Voting

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SSEAC May 25-26, 2009

Ranking approach

Research strategy:

- to predict or understand behavior (e.g. manage conflicts) one needs to know the goals and beliefs of the parties involved
- the goals are preferred states of the world
- given the goals, the beliefs restrict the action possibilities to those believed to lead to those goals
- assuming that goals are many and resources limited, the principle of rationality calls for acts that lead to goal achievement in a rational manner (e.g. with minimum associated costs)
- prediction: the actors will resort to those acts that will lead to their goals in a rational way
- design principle: look for those mechanisms that result in desired outcomes as game-theoretic equilibria

Representing goals and rationality

Definition

Rationality. A decision maker is rational if – when confronted with the choice between state a and state b – he/she will choose state a iff he/she prefers a to b.

Theorem

(Harsanyi 1977). Suppose that a preference relation is complete and transitive over the outcomes and that for each alternative state, the inferior and superior states constitute closed sets. Then the preference can be represented by a utility function.

Analogous representation theorems have been proven for risky and uncertain prospects.

But cyclic preferences may make sense

Example

Three universities A, B and C are being compared along three criteria: (i) research output (scholarly publications), (ii) teaching output (degrees), (iii) external impact (expert assignments, media visibility, R& D projects, etc.)

crit. (i)	crit. (ii)	crit. (iii)
A	B	C
B	C	A
C	A	B

Cycle: $A \succ B \succ C \succ A \succ \ldots$.

Asking for individual choice functions

Consider the following (plausibility) conditions on collective choices based on individual choice functions

- citizen sovereignty: for any alternative x, there exists a set of individual choice function values so that x is will be elected,
- choice-set monotonicity: if x is elected under some profile of individual choices, then x should also be elected if more individuals include x in their individual choices
- neutrality
- anonymity, and
- choice-set Pareto: if all individuals include x in their individual choice sets, then the aggregation rule includes x as well , and if no voter includes y in their individual choice set, then y is not included in the collective choice.

Same requirements for individuals and collectivities

In social welfare functions the aim is to impose the same formal properties on the aggregation rule as on the individual opinions: completeness and transitivity of preference relations. Surely some conditions have to be imposed on choice set aggregation rules to distinguish reasonable from unreasonable ones. Two conditions:

- Chernoff's condition: if an alternative is among winners in a large set of alternatives, it should also be among the winners in every subset it belongs to.
- Concordance: suppose that the winners in two subsets of alternatives have some common alternatives. Then the rule is concordant if these common alternatives are also among the winners in the union of the two subsets.

We can insist that Chernoff and concordance be satisfied by both functions.

Two reasonable rules fail

Rule 1: whenever an alternative is included in the choice sets of a majority of voters, it will be elected.
Rule 2 (plurality): whichever alternative is included in more numerous choice sets than the other alternatives, is elected.

alt. set	ind. choice sets			rule 1	rule 2
	ind.1	ind. 2	ind. 3		
$\{x, y, z\}$	$\{x\}$	$\{z\}$	$\{y\}$	$\emptyset$	$\{x, y, z\}$
$\{x, y\}$	$\{x\}$	$\{x\}$	$\{y\}$	$\{x\}$	$\{x\}$
$\{x, z\}$	$\{x\}$	$\{z\}$	$\{x\}$	$\{x\}$	$\{x\}$
$\{y, z\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\{y\}$	$\{y\}$

Concordance is not satisfied by rule 1, since x is chosen from $\{x, y\}$ and $\{x, z\}$, but not from $\{x, y, z\}$. Rule 2 fails on Chernoff since z is in the choice set from $\{x, y, z\}$, but from $\{x, z\}$. N.B.: plurality (but not majority) fails on choice-set monotonicity.

Individual tournaments as input

Tournament:

- complete and asymmetric relation
- can (and often is) formed from preference profiles
- is less demanding than assuming individual choice functions
- can be more demanding than choice functions (when the latter are non-resolute)

Choices from tournaments:

- uncovered set
- Copeland winners
- Banks' set

From individual to collective opinions

Rules (e.g.):

1. Given the individual $k \times k$ tournaments, construct the corresponding collective one of the same dimension by inserting 1 to position (i, j) if more than $n/2$ individual have 1 in the (i, j) position. Otherwise, insert 0 to this position. The row sums then indicate the Copeland scores. Rows with sum equal to zero correspond to Condorcet losers, those with sums equal to $k - 1$ Condorcet winners. Uncovered and Bank's sets can be computed as well (the latter, though, is computationally hard). Also Dodgson scores can determined.
2. Construct the collective opinion matrix as an outranking matrix where the entry in the (i, j) position equals the number of individuals with 1 in the (i, j) position. The row sums then indicate the "Borda scores". Max-min scores can also be determined.

“Theorists’ fixation”

Hillinger (2004):

> *... a new ‘paradox of voting’: It is theorists’ fixation on a context dependent and ordinal preference scale; the most primitive scale imaginable and the mother of all paradoxes.*

Utilitarian voting

Definition

Hillinger: Let P be a strict preference relation of a voter and let P assign the set of candidates into disjoint subsets $A_1, \ldots, A_K, K \geq 1$ such that the voter is indifferent between candidates in the same subset and strictly prefers $a_i \in A_i$ to candidate $a_j \in A_j$ iff $i > j$.K is given independently of the number of candidates. For a given K, the voter is asked to assign to each candidate one of the numbers $x_0, x_0 + 1, \ldots, x_0 + K - 1$. The utilitarian voting winner is the alternative with the largest arithmetic mean or sum of assigned numbers.

Names and precursors

Synonyms:

- Bentham's method (Riker 1982)
- range voting (W. Young)
- utilitarian voting (Hillinger)
- evaluative voting (Hillinger)

N.B. cumulative voting is not equivalent with the systems above since it allows for a flexible upper bound on the votes to be given to each candidate.

Majoritarian judgment

The process:

1. each voter gives each candidate a grade (ordinal)
2. the median grade of each candidate is determined
3. the winner is the candidate with the highest median grade
4. a specific tie-breaking rule is defined

Grades: $A \prec B \prec C \prec \ldots$.

Assessment by Felsenthal and Machover

Some advantages:

- IIA or Chernoff
- monotonicity
- immunity to cloning

MJ assessment, cont'd

Some disadvantages:

- inconsistency (violation of reinforcement)
- vulnerability to no-show paradox:

candidate	voters 1 - 3	voter 4	voters 5 - 7
x	A	D	E
y	B	C	F

  x wins, but if voters 1 and 2 abstain y – their favorite – wins.
- Condorcet loser might get elected:

candidate	voter 1	voter 2	voter 3
x	B	C	F
y	A	D	E
z	C	C	G

The Lehrer-Wagner system; basic version

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Weights:

$$W = \begin{bmatrix} w_{11} & \cdots & w_{1k} \\ w_{21} & \cdots & w_{2k} \\ \cdots & \cdots & \cdots \\ w_{n1} & \cdots & w_{nk} \end{bmatrix}$$

with

- $[w_{ij} \in [0, 1]$, and
- $sum_j w_{ij} = 1, \forall i, j.$

Basic version: how does it work?

Under very general conditions the sequence

$$W, W^2, W^3, \dots,$$

converges to a matrix with identical rows. Denote a row of this matrix by C. These are consensual weights assigned to each voter. In the basic vector the system simply picks the person with the largest weight to make the choice on behalf of the collectivity.

The Lehrer-Wagner system; second version

Utilities:

$$U = \begin{bmatrix} u_{11} & \cdots & u_{1k} \\ u_{21} & \cdots & u_{2k} \\ \cdots & \cdots & \cdots \\ u_{n1} & \cdots & u_{nk} \end{bmatrix}$$

Second version: how does it work?

CU indicates the weighted utilities of each candidate or alternative. The choice is the alternative with the largest weighted utility.

Some properties of the basic version

- it is monotonic in preferences when weight matrices are fixed
- it is Pareto optimal
- it does not satisfy Condorcet winner criterion: Let

$$U = \begin{bmatrix} 1.0 & 0.8 & 0.5 \\ 0.5 & 1.0 & 0.7 \\ 0.5 & 0.6 & 1.0 \end{bmatrix}$$

and $C = [0.1, 0.2, 0.7]$. Then $CA = [0.55, 0.70, 0.89]$, i.e. third alternative is chosen, not the second (the Condorcet winner).

Some properties, cont'd

- it does not exclude Condorcet losers:

$$U = \begin{bmatrix} 1.0 & 0.6 & 0.5 \\ 0.5 & 1.0 & 0.6 \\ 0.5 & 0.6 & 1.0 \end{bmatrix}$$

  and $C = [0.7, 0.2, 0.1]$. Then $CA = [0.85, 0.68, 0.57]$, i.e. the Condorcet loser is chosen.
- it is inconsistent, provided that some changes in weight matrices be made to make convergence possible.

Medians and geometric averages

Instead of taking arithmetic averages of the Borda points $b_i(x)$ or the numbers $n(x, y)$, we could consider the medians or geometric averages of these numbers. Given an m -dimensional vector y, the median of $M(y)$ of y is calculated as follows. Index the coordinates of y by the numbers $1, \ldots, m$ so that $y_i \leq y_j$ if $i \leq j$, $i, j \in \{1, \ldots, m\}$. If m is odd, then $m = 2k + 1$ for some $k \in \mathbb{N}$, and $M(y) = y_{k+1}$. If m is even, then $m = 2k$ for some $k \in \mathbb{N}$, and $M(y) = (y_k + y_{k+1})/2$.
The geometric average $G(y)$ of the coordinates of y is $G(y) = \sqrt[m]{\prod_i y_i}$, if all numbers y_i are nonnegative.

Four more Borda variations

For all problems $A = (N, X, R^N)$, let $b(x)$ be the vector $(b_i(x))_{i \in N}$ and let $n(x)$ be the vector $(n(x, y))_{y \in X}$. Applying the median, we define the rules f^{Mb} and f^{Mn} by

- $f^{Mb}(A)(x) = M(b(x))$, and
- $f^{Mn}(A)(x) = M(n(x))$, for all $x \in X$, for all problems $A = (N, X, R^N)$.

Applying the geometric average, we define the rules f^{Gb} and f^{Gn} by

- $f^{Gb}(A)(x) = G(b(x))$, and
- $f^{Gn}(A)(x) = G(n(x))$, for all $x \in X$, for all problems $A = (N, X, R^N)$.

f^{Mb} fails on Condorcet loser

Example

1 voter	ABCDE
1 voter	CBADE
1 voter	DBEAC
1 voter	EBADC
1 voter	ADCEB
1 voter	ACDEB
1 voter	CDEAB

Both f^{Gb} and f^{Gn} may elect a Condorcet loser

Example

1 voter	ABCDEFGHIJ
1 voter	JABCDFEGHI
1 voter	IJABCFDEGH
1 voter	HIJABFCDEG
1 voter	GHIJAFBCDE
1 voter	EGHIJFABCD
1 voter	DEGHIFJABC
1 voter	CDEGHFIJAB
1 voter	BCDEGFHIJA

f^{Mb} is not consistent

Example

1 voter	ABCD
1 voter	ABDC
1 voter	BACD
1 voter	BCDA
1 voter	ABCD
1 voter	ABDC
1 voter	DABC
1 voter	CDBA
1 voter	CDAB

f^{Mn} is not consistent

Example

1 voter	ACBD
2 voters	BACD

1 voter	ACBD
2 voters	CABD

Partial summary

criterion method	Condorcet-loser	consistency
BC	yes	yes
f^{Mb}	no	no
f^{Mn}	yes	no
f^{Gb}	no	yes
f^{Gn}	no	yes
maximin	no	no
maximax	yes	no
Litvak	no	no

Is plurality runoff an improvement over plurality?

Example

6 voters	*5 voters*	*4 voters*	*2 voters*
A	C	B	B
B	A	C	A
C	B	A	C

Or Nanson's system over Borda's?

Example

30	21	20	12	12	5
C	B	A	B	A	A
A	D	B	A	C	C
D	C	D	C	B	D
B	A	C	D	D	B

How about a hybrid system: Black?

How does it work? Pick the Condorcet winner. If none exists, choose the Borda winner.

Properties:

- Satisfies Cordorcet criteria
- Is monotonic
- Is inconsistent

Example

4 voters	*3 voters*	*3 voters*	*2 voters*	*2 voters*
A	B	A	B	C
B	C	B	C	A
C	A	C	A	B

Conclusions

- alternatives to ranking assumption are worth pursuing since the social choice desiderata are dependent on voter input assumptions
- the notion of “thin rationality” is ranking-based. Analogous notions are yet to be defined in other than ranking environments
- present voting systems call for – and thus utilize – very little information on voter views
- systems enabling the voters to reveal their opinions in richer ways might activate voters especially in democracies with dwindling turnouts
- some such systems have already been proposed and assessed

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