# ON THE MANIPULABILITY OF APPROVAL VOTING

Hans Peters, University of Maastricht

This presentation is based on research in progress joint with Souvik Roy, Storcken; and Bram Driesen (simulations)

Overview

1. Introduction
2. Approval voting
3. $k$-Approval voting
4. Conclusions

# 1. Introduction (1)

- Point of departure: any reasonable voting rule is manipulable (Gibbard 1973, Satterthwaite 1975).
- Accepting this fact, we ask: what is the extent of manipulability of a voting rule? Which rule is 'best' in this respect?
- We concentrate on approval voting and $k$-approval voting.
- We characterize the (non-)manipulable preference profiles and give some analytical and numerical (simulation) results.

Related early works:

Kelly (1988, 1989, 1993); Fristrup and Keiding (1998); Aleskerov and Kurbanov (1999); Slinko (2002); Maus, Peters, and Storcken (2007a,b,c,d).

# 1. Introduction (2)

Basic model

$N = \{1, \ldots, n\}$, $n \geq 2$: set of *voters*

$A$, $3 \leq |A| < \infty$: set of *alternatives*

$L$: set of *preferences* (= linear orderings) over $A$

$F : L^N \to A$ or $2^A \setminus \{\emptyset\}$: *social choice function* (SCF) or *correspondence* (SCC)

SCF $F$ is *manipulable* at profile $R^N \in L^N$ by voter $i \in N$ if there is a $Q^i \in L$ such that

$$F(Q^i, R^{N\setminus\{i\}})\, R^i\, F(R^N), \quad F(Q^i, R^{N\setminus\{i\}}) \neq F(R^N).$$

$F$ is *non-manipulable* (or *strategy-proof*) if it is not manipulable at any profile by any voter.

$F$ is *dictatorial* if there is $i \in N$ such that $F(R^N)$ is the top alternative of $R^i$ for every $R^N \in L^N$.

## 1. Introduction (3)

Theorem (Gibbard 1973, Satterthwaite, 1975)

Any surjective non-manipulable social choice function is dictatorial.

Drawbacks of manipulability

(i) Social choice function loses attractive properties.
(ii) Ethical objections.
(iii) Cost of manipulation.

An example

| Voter | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|
| 1 | 5 | 1 | 3 | 2 | 4 |
| 2 | 1 | 2 | 3 | 4 | 5 |
| 3 | 3 | 4 | 5 | 2 | 1 |

Borda rule results in $a_3$, which indeed seems a good compromise. If voter 1 change his scores to 5,3,1,2,4 then $a_5$ results.

# 1. Introduction (4)

Reactions to the Gibbard-Satterthwaite result

- Restricted domains, e.g., single-peakedness (Black, 1948; Dummett and Farquharson, 1961; Blin and Satterthwaite, 1976; Moulin, 1980, etc.). Huge literature.
- Probabilistic mechanisms, not very 'successful', e.g. Gibbard (1977), etc.
- Exact and strong consistency (Peleg, 1978, etc.).
- Minimal manipulability, initiated by Kelly (1988): characterize manipulable profiles, find social choice functions or correspondence that are 'minimally manipulable' within a certain class of mechanisms.

We follow the last approach and concentrate on approval and $k$-approval voting.

## 2. Approval voting (1)

Definition

Within our basic model, *approval voting* is a correspondence

$$AP : (L \times \{1, \dots, m\})^N \to 2^A \setminus \{\emptyset\}.$$

If player $i$ reports $(R^i, k^i)$ then the top $k^i$ alternatives of $R^i$ each get 1 point; sum over all players; then $AP\left((R^i, k^i)_{i \in N}\right)$ is the subset of $A$ of alternatives with maximal number of points.

Preference extension

Since approval voting (as well as, later on, $k$-approval voting and other scoring rules) is a correspondence, we need to extend preferences from alternatives to sets. We will consider several such extensions: worst comparison, best comparison, stochastic comparison.

## 2. Approval voting (2)

Dichotomous preferences

If voter $i$ with report $(R^i, k^i)$ is indifferent between his top $k^i$ alternatives ('acceptable alternatives') and also between his bottom $m-k$ alternatives ('non-acceptable alternatives'), then his preference is *dichotomous.*

Brams and Fishburn (1983) show that under mild conditions on preference extension approval voting is non-manipulable if preferences are dichotomous.

This result also holds under the preference extensions that we will consider.

If, however, preferences can be are more refined then this is no longer true even while reports are still 'dichotomous'.

## 2. Approval voting (3)

Preference extensions

Throughout, we consider three different preference extensions.

Let $i \in N$ and $R^i \in L$.

*Worst comparison*: $\succeq^w_{R^i}$

For all non-empty sets $B, C \in 2^A$, $B \succeq^w_{R^i} C$ whenever the worst element of $B$ according to $R^i$ is at least as good as the worst element of $C$ according to $R^i$.

*Best comparison*: $\succeq^b_{R^i}$

For all non-empty sets $B, C \in 2^A$, $B \succeq^b_{R^i} C$ whenever the best element of $B$ according to $R^i$ is at least as good as the best element of $C$ according to $R^i$.

## 2. Approval voting (4)

*Stochastic comparison:* $\succeq^s_{R^i}$

For all non-empty sets $B, C \in 2^A$, $B \succeq^s_{R^i} C$ whenever the equal chance lottery on $B$ weakly stochastically dominates the equal chance lottery on $C$. (I.e., its expected utility is at least as large under any utility representation of $R^i$.)

(Cf. Barberà, Dutta, Sen, 2001)

(Non-)manipulability

Voter $i$ can *manipulate* social choice correspondence $F$ at $R^N$ under preference extension $\succeq_{R^i}$ if he has a preference $Q^i$ such that $F(Q^i, R^{N\setminus\{i\}}) \succ_{R^i} F(R^N)$.

Here, $\succ_{R^i}$ is the asymmetric part of $\succeq_{R^i}$.

For approval voting: $(Q^i, \ell^i)$ instead of $(R^i, k^i)$.

$F$ is *non-manipulable* if no voter can manipulate.

# 2. Approval voting (5)

An example

Six voters $1, \ldots, 6$, four alternatives $a, b, c, d$. We consider manipulation by voter 1.

(1) The votes from $2, \ldots, 6$ add up to $4, 4, 3, 2$ for $a, b, c, d$, respectively.
Preference of voter 1: $cab|d$.
Truthful voting (i.e., $R^1 = cabd$, $k^1 = 3$) results in $\{a, b\}$.
Voting only for $a$ and $c$: $\{a\}$, improves by worst and by stochastic comparison.
Voting only for $c$: $\{a, b, c\}$, which is better by best comparison.

## 2. Approval voting (6)

(2) Now the votes cast by $2, \ldots, 6$ add up to $2, 4, 2, 4$.
Voter 1 has preference $ca|bd$.
Truthful voting results in $\{b, d\}$.
Voting for $b$, $a$ and $c$ results in $\{b\}$, which is better by worst and stochastic comparison.

(3) The votes cast by $2, \ldots, 6$ add up to $3, 4, 2, 2$.
Voter 1 has preference $c|abd$.
Truthful voting results in $\{b\}$.
Voting for $a$ and $c$ results in $\{a, b\}$, which is better by best comparison.

In all these examples voter 1 still votes for a top ranked set of candidates, even if he manipulates. Nevertheless, he may sometimes not vote for a candidate even if he finds that candidate acceptable, or vote for a candidate even if he finds that candidate not acceptable.

## 2. Approval voting (7)

In our work (in progress) we characterize all profiles of preferences that are non-manipulable under worst, best, or stochastic dominance comparison. These descriptions are quite technical, but useful for comparison with other voting rules and for simulation purposes. Recall that we consider only strict preferences in this presentation.

Theorem worst comparison approval voting

The approval rule is non-manipulable at some profile under worst comparison if and only if for each agent $i \in N$ at least one of the following two statements holds:

(a) Exactly one alternative among his $k^i$ first ranked alternatives is also among the winners without that agent's vote.

(b) There is exactly one winner without that agent's vote. ■

## 2. Approval voting (8)

Theorem best comparison approval voting

The approval rule is non-manipulable at some profile under best comparison if and only if for each agent $i \in N$ at least one of the following two statements holds:
(a) The best alternative among the winners without that agent's vote is preferred to the almost (= one vote less) winners without that agent's vote.
(b) None of his $k^i$ best alternatives is among the winners without his vote, some of them are among the almost winners without that agent's vote. ■

There is also a theorem for stochastic comparison but this is (more) technical. See the paper (forthcoming).

Roughly: non-manipulability is more demanding under stochastic comparison than under best and worst comparison.

## 2. Approval voting (9)

Some simulation results: rounded off percentages of *non-manipulable* profiles, 100,000 profiles

| **n = 2** | $m$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| | worst comp | 56 | 45 | 38 | 33 | 30 | 27 | 25 | 23 |
| | best comp | 65 | 59 | 55 | 52 | 49 | 48 | 46 | 45 |
| | stoch comp | 41 | 37 | 34 | 31 | 30 | 28 | 27 | 25 |

| $n = 6$ | $m$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| | worst comp | 67 | **61** | 57 | 54 | 52 | 50 | 48 | 47 |
| | best comp | 51 | **40** | 33 | 29 | 25 | 22 | 20 | 18 |
| | stoch comp | 28 | **20** | 16 | 13 | 11 | 10 | 09 | 08 |

| $n = 10$ | $m$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| | worst comp | 73 | 69 | 65 | 62 | 60 | 59 | 57 | 56 |
| | best comp | 54 | 43 | 37 | 32 | 28 | 25 | 23 | 21 |
| | stoch comp | 35 | 27 | 22 | 19 | 16 | 15 | 13 | 12 |

## 2. Approval voting (10)

<u>Some simulation results</u>: rounded off percentages of *non-manipulable* profiles, 100,000 profiles

| $\mathbf{n = 3}$ | $m$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| | worst comp | 57 | 49 | 44 | 41 | 38 | 36 | 34 | 32 |
| | best comp | 56 | 47 | 41 | 38 | 34 | 32 | 30 | 28 |
| | stoch comp | 25 | 20 | 17 | 16 | 15 | 14 | 14 | 13 |

| $n = 6$ | $m$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| | worst comp | 67 | **61** | 57 | 54 | 52 | 50 | 48 | 47 |
| | best comp | 51 | **40** | 33 | 29 | 25 | 22 | 20 | 18 |
| | stoch comp | 28 | **20** | 16 | 13 | 11 | 10 | 09 | 08 |

| $n = 10$ | $m$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| | worst comp | 73 | 69 | 65 | 62 | 60 | 59 | 57 | 56 |
| | best comp | 54 | 43 | 37 | 32 | 28 | 25 | 23 | 21 |
| | stoch comp | 35 | 27 | 22 | 19 | 16 | 15 | 13 | 12 |

## 2. Approval voting (11)

Some (very preliminary) conclusions:

- Manipulability increases with the number of alternatives and decreases with the number of agents – but there are exceptions when $n$ is relatively low.
- Manipulability is largest for stochastic dominance comparison and lowest for worst comparison – with an exception for $n = 2$.
- Simulations may be helpful to obtain more analytical results.

## 3. $k$-Approval voting (1)

Definition

For $k \in \{1, \ldots, m-1\}$, *$k$-approval voting* is a correspondence

$$AP_k : L^N \to 2^A \setminus \{\emptyset\}.$$

If player $i$ reports $R^i$ then the top $k$ alternatives of $R^i$ each get 1 point; sum over all players; then $AP_k(R^N)$ is the subset of $A$ of alternatives with maximal number of points.

For instance, for $k = 1$ we have the plurality rule.

$k$-approval voting is a scoring rule. It is less flexible than approval voting (voters report less preference information). Is it also less manipulable?

We consider the same preference extensions as before.

# 3. $k$-Approval voting (2)

Theorem worst comparison approval voting

The $k$-approval rule is non-manipulable at some profile under worst comparison if and only if for each agent $i \in N$ at least one of the following three statements holds:
(a) Exactly one alternative among his $k$ first ranked alternatives is also among the winners without that agent's vote.
(b) There is exactly one winner without that agent's vote.
(c) His $m-k$ worst alternatives are a strict subset of the set of winners without that agent's vote. ■

We have also theorems for best and stochastic comparison, see paper. These results can be used for comparison of different $k$ values, for comparison with other scoring rules, and for comparison with approval voting.

## 3. $k$-Approval voting (3)

Two voters ($n = 2$)

- $k = 1$, plurality voting, strategy-proof for $n = 2$ under all three preference comparisons. Not unambiguously attractive, e.g., profile $(ab \ldots c, cb \ldots a)$.

- $k \geq 2$, worst comparison.

Number of non-manipulable profiles increases between 2 and $k^* \approx \sqrt{m}$, decreases between $k^*$ and $\frac{1}{2}(m-1)$, and increases up to $m-1$. The optimal (minimally manipulable) value is $m-1$.

- $k \geq 2$, best comparison.

Is not manipulable for $k > m/2$. For $k < m/2$ the optimal value is $k = 2$.

- $k \geq 2$, stochastic comparison.

For $k \leq m/2$ the optimal value is 2. For $k > m/2$ the optimal value is $m-1$. The overall optimal value is $k = 2$.

## 3. $k$-Approval voting (4)

- Call a rule $F$ *citizen sovereign* if for each $a \in A$ there is $R^N \in L^N$ with $F(R^N) = \{a\}$.

$k$-approval voting is citizen sovereign if and only if $k \leq m/2$. Is an argument for considering $k \leq m/2$.

More than 2 voters ($n > 2$)

- Conjecture: if $n \to \infty$ then $AP_k$ with $k = m/2$ (or $k \in \{\frac{m-1}{2}, \frac{m+1}{2}\}$) is optimal, even *among all scoring rules*.

Intuition: the score vector has maximal variance for $k = m/2$, and so the expected difference between the winner and the second best becomes maximal.

Result suggests that for a relatively large number of agents $k = m/2$ might be optimal (simulation results later on).

(See also Pritchard and Wilson, 2009, in a different context, but same intuition.)

# 3. $k$-Approval voting (5)

Some simulation results

$m = 6,\ n = 10$, % non-manipulable profiles (100,000 trials)

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| worst comp | 69 | 70 | 68 | 64 | 56 |
| best comp | 45 | 56 | 52 | 40 | 51 |
| stoch comp | 29 | 36 | 31 | 27 | 07 |

$m = 10,\ n = 25$, % non-manipulable profiles (1000,000 trials)

| $k$ | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| worst comp | 67 | 74 | 74 | 70 | 49 |
| best comp | 34 | 50 | 50 | 41 | 54 |
| stoch comp | 27 | 38 | 38 | 30 | 03 |

## 3. $k$-Approval voting (6)

For our earlier example with 6 voters, 4 alternatives, % of non-manipulable profiles.

| | $APP$ | $APP_2$ | $APP_3$ |
|---|---|---|---|
| worst comp | 61 | 72 | 79 |
| best comp | 40 | 59 | 57 |
| stoch comp | 20 | 48 | 36 |

Is in line with what we found for $n = 2$, and indicates that $k$-approval voting is less manipulable than approval voting. Other simulation results indicate that this is true more generally, in particular for best and stochastic comparison.

## 3. Conclusions

• Work in progress, analytical results difficult to obtain; simulations may be useful here.

• Nevertheless: $k$-approval voting might be a good alternative for approval voting from the point of view of (minimal) manipulability. For many agents perhaps $k = m/2$.

• Characterizations of non-manipulable profiles may also be useful when studying partial culture.

• Results (or indications of results) apply in particular to situations as for instance voting within scientific communities: relatively few candidates and many more but still relatively few voters.

• Theoretically, individual manipulation becomes irrelevant in large voting situations. Still, one may expect groups of voters to have similar preferences (partial culture) and to strategically vote in the same way. (Cf. Pritchard and Wilson, 2009).