A DISTANCE-BASED EXTENSION OF THE MAJORITY JUDGEMENT VOTING SYSTEM

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ABSTRACT. It is common knowledge that the political voting systems suffer inconsistencies and paradoxes such that Arrow has shown in his well-known Impossibility Theorem. Recently Balinski and Laraki have introduced a new voting system called Majority Judgement (MJ) which tries to solve some of these limitations. In MJ voters have to asses the candidates through linguistic terms belonging to a common language. From this information, MJ assigns as the collective assessment the lower median of the individual assessments and it considers a sequential tie-breaking method for ranking the candidates. The present paper provides an extension of MJ focused to reduce some of the drawbacks that have been detected in MJ by several authors. The model assigns as the collective assessment a label that minimizes the distance to the individual assessments. In addition, we propose a new tie-breaking method also based on distances.

1. INTRODUCTION

Social Choice Theory shows that there does not exist a completely acceptable voting system for electing and ranking alternatives. The well-known Arrow Impossibility Theorem [1] proves with mathematic certainty that no voting system simultaneously fulfills certain desirable properties[1]. Recently Balinski and Laraki [2, 4, 5] have proposed a voting system called Majority Judgement (MJ) which tries to avoid these unsatisfactory results and allows the voters to assess the alternatives through linguistic labels, as *Excellent*, *Very good*, *Good*, ..., instead of rank order the alternatives. Among all the individual assessments given by the voters, MJ chooses the median as the collective assessment. Balinski and Laraki also describe a tie-breaking process which compares the number of labels above the collective assessment and those below of it. These authors also have an experimental analysis of MJ [3] carried out in Orsay during the 2007 French presidential election. In that paper the authors show some interesting properties of MJ and they advocate that this voting system is easily implemented and that it avoids the necessity for a second round of voting. Desirable properties and advantages have been attributed to MJ against the classical Arrow framework of preferences' aggregation. Among them are the possibility that voters show more faithfully and properly their opinions than in the conventional voting systems, anonymity, neutrality, independence of irrelevant alternatives, etc. However, some authors (see Felsenthal and Machover [6], García-Lapresta and Martínez-Panero [7] and Smith [9]) have shown several paradoxes and inconsistencies of MJ. In this paper we propose an extension of MJ which diminishes

[1]Any voting rule that generates a collective weak order from every profile of weak orders, and satisfies independence of irrelevant alternatives and unanimity is necessarily dictatorial, insofar as there are at least three alternatives and three voters.

some of the MJ inconveniences. The approach of the paper is distance-based, both for generating a collective assessment of each alternative and in the tie-breaking process that provides a weak order on the set of alternatives. As in MJ we consider that individuals assess the alternatives through linguistic labels and we propose as the collective assessment a label that minimizes the distance to the individual assessments. These distances between linguistic labels are induced by a metric of the parameterized Minkowski family. Depending on the specific metric we use, the discrepancies between the collective and the individual assessments are weighted in a different manner, and the corresponding outcome can be different. The paper is organized as follows. In Section 2, the MJ voting system is formally explained. Section 3 introduces our proposal, within a distance-based approach. Specifically, the election of the collective assessment for each alternative and the tie-breaking method are introduced. In Section 4 we include two illustrative examples showing the influence of the metric used in the proposed method and its differences with respect to MJ and Range Voting (Smith [9]). Finally, in Section 5 we collect some conclusions.

2. Majority Judgement

We consider[2] a finite set of voters $V = \{1, \dots, m\}$, with $m \geq 2$, who evaluate a finite set of alternatives $X = \{x_1, \dots, x_n\}$, with $n \geq 2$. Each alternative is assessed by each voter through a linguistic term belonging to an ordered finite scale $L = \{l_1, \dots, l_g\}$, with $l_1 < \cdots < l_g$ and granularity $g \geq 2$. Each voter assesses the alternatives in an independent way and these assessments are collected by a matrix $\left(v_j^i\right)$, where $v_j^i \in L$ is the assessment that the voter i gives to the alternative x_j. MJ chooses for each alternative the median of the individual assessment as the collective assessment. To be precise, the single median when the number of voters is odd and the lower median in the case that the number of voters is even. We denote with $l(x_j)$ the collective assessment of the alternative x_j. Given that several alternatives might share the same collective assessment, Balinski and Laraki [2] propose a sequential tie-breaking process. This can be described through the following terms (see García-Lapresta and Martínez-Panero [7]):

$$N^+(x_j) = \#\{i \in V \mid v_j^i > l(x_j)\}\,, \quad N^-(x_j) = \#\{i \in V \mid v_j^i < l(x_j)\}$$

and

$$t(x_j) = \begin{cases} -1, & \text{if } N^+(x_j) < N^-(x_j), \\ 0, & \text{if } N^+(x_j) = N^-(x_j), \\ 1, & \text{if } N^+(x_j) > N^-(x_j). \end{cases}$$

Taking into account the collective assessments and the previous indices, we define a weak order[3] $\succeq$ on X in the following way: $x_j \succeq x_k$ if and only if one of the following conditions hold:

(1) $l(x_j) > l(x_k)$.
(2) $l(x_j) = l(x_k)$ and $t(x_j) > t(x_k)$.
(3) $l(x_j) = l(x_k)$, $t(x_j) = t(x_k) = 1$ and $N^+(x_j) > N^+(x_k)$.

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[2] The current notation is similar to the one introduced by García-Lapresta and Martínez-Panero [7]. This allows us to describe the MJ process, presented by Balinski and Laraki [2], in a more precise way.

[3] A *weak order* (or *complete preorder)* is a complete and transitive binary relation.

(4) $l(x_j) = l(x_k)$, $t(x_j) = t(x_k) = 1$, $N^+(x_j) = N^+(x_k)$ and $N^-(x_j) \leq N^-(x_k)$.
(5) $l(x_j) = l(x_k)$, $t(x_j) = t(x_k) = 0$ and $m - N^+(x_j) - N^-(x_j) \geq m - N^+(x_k) - N^-(x_k)$.
(6) $l(x_j) = l(x_k)$, $t(x_j) = t(x_k) = -1$ and $N^-(x_j) < N^-(x_k)$.
(7) $l(x_j) = l(x_k)$, $t(x_j) = t(x_k) = -1$, $N^-(x_j) = N^-(x_k)$ and $N^+(x_j) \geq N^+(x_k)$.

The asymmetric and symmetric parts of $\succeq$ are defined in the usual way:

$$x_j \succ x_k \Leftrightarrow \text{ not } x_k \succeq x_j$$
$$x_j \sim x_k \Leftrightarrow (x_j \succeq x_k \text{ and } x_k \succeq x_j).$$

Next an example of how MJ works is shown.

Example 1. Consider three alternatives x_1, x_2 and x_3 that are evaluated by seven voters through a set of six linguistic terms $L = \{l_1, \ldots, l_6\}$, the same set used in MJ [3], whose meaning is shown in Table 1. The assessments obtained for each alternative are collected and ranked from the lowest to the highest in Table 2. For ranking the three alternatives, first we take the median of the individual assessments, that will be the collective assessment for each one of the mentioned alternatives: $l(x_1) = l_5$, $l(x_2) = l_4$ and $l(x_3) = l_4$. Given that x_1 has the best collective assessment, it will be the one ranked in first place. However, the alternatives x_2 and x_3 share the same collective assessment, we need to turn to the tie-breaking process, where we obtain $N^+(x_2) = 2$, $N^-(x_2) = 1$ and $t(x_2) = 1$; $N^+(x_3) = 3$, $N^-(x_3) = 2$ and $t(x_3) = 1$. Since both alternatives have the same t ($t(x_2) = t(x_3) = 1$), we should compare their N^+: $N^+(x_2) = 2 < 3 = N^+(x_3)$. Therefore, the alternative x_3 defeats the alternative x_2, and the final order is $x_1 \succ x_3 \succ x_2$.

TABLE 1. Meaning of the linguistic terms

l_1	l_2	l_3	l_4	l_5	l_6
To reject	*Poor*	*Acceptable*	*Good*	*Very good*	*Excellent*

TABLE 2. Assessments of Example 1

x_1	l_1	l_1	l_3	l_5	l_5	l_5	l_6
x_2	l_1	l_4	l_4	l_4	l_4	l_5	l_6
x_3	l_1	l_3	l_4	l_4	l_5	l_5	l_5

3. Distance-based method

In this section the alternative method to MJ that we propose through a distance-based approach is introduced. The first step for ranking the alternatives is to assign a collective assessment $l(x_j) \in L$ to each alternative $x_j \in X$. For its calculation, the vectors $(v_j^1, \ldots, v_j^m)$ that collect all the individual assessments for each alternative $x_j \in X$ are taken into account. The proposal, that is detailed below, involves how to choose a $l(x_j) \in L$ that minimizes the distance between the vector of individual assessments $(v_j^1, \ldots, v_j^m)$ and the vector $(l(x_j), \ldots, l(x_j)) \in L^m$. The election of that term is performed in an independent way for each alternative. This guarantees the fulfillment of the *independence of irrelevant alternatives principle*[4]. Once a collective assessment $l(x_j)$ has been associated with each alternative $x_j \in X$, we rank the alternatives according to the ordering of L. Given the possible existence of ties, we also propose a sequential tie-breaking process based on the difference between the distance of $l(x_j)$ to the assessments higher than $l(x_j)$ and the distance of $l(x_j)$ to the assessments lower than $l(x_j)$.

3.1. **Distances.** A *distance* or *metric* on $\mathbb{R}^m$ is a mapping $d : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ that fulfills the following conditions for all $(a_1, \ldots, a_m), (b_1, \ldots, b_m), (c_1, \ldots, c_m) \in \mathbb{R}^m$:

(1) $d((a_1, \ldots, a_m), (b_1, \ldots, b_m)) \geq 0$.
(2) $d((a_1, \ldots, a_m), (b_1, \ldots, b_m)) = 0 \Leftrightarrow (a_1, \ldots, a_m) = (b_1, \ldots, b_m)$.
(3) $d((a_1, \ldots, a_m), (b_1, \ldots, b_m)) = d((b_1, \ldots, b_m), (a_1, \ldots, a_m))$.
(4) $d((a_1, \ldots, a_m), (b_1, \ldots, b_m)) \leq$
$d((a_1, \ldots, a_m), (c_1, \ldots, c_m)) + d((c_1, \ldots, c_m), (b_1, \ldots, b_m))$.

Given a distance $d : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$, the *distance on* L^m *induced by* d is the mapping $\bar{d} : L^m \times L^m \longrightarrow \mathbb{R}$ defined by

$$\bar{d}((l_{a_1}, \ldots, l_{a_m}), ((l_{b_1}, \ldots, l_{b_m})) = d((a_1, \ldots, a_m), (b_1, \ldots, b_m)).$$

An important class of distances in $\mathbb{R}^m$ is constituted by the family of *Minkowski distances* $\{d_p \mid p \geq 1\}$, which are defined by

$$d_p((a_1, \ldots, a_m), (b_1, \ldots, b_m)) = \left(\sum_{i=1}^{m} |a_i - b_i|^p \right)^{\frac{1}{p}},$$

for all $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_m) \in \mathbb{R}^m$. We choose this family due to the fact that it is parameterized and it includes from the well-known *Manhattan* ($p = 1$) and *Euclidean* ($p = 2$) distances, to the limit case, the *Chebyshev* distance ($p = \infty$). The possibility of choosing among different values of $p \in (1, \infty)$ gives us a very flexible method, and we can choose the most appropriate p according to the objectives we want to achieve with the election. Given a Minkowski distance on $\mathbb{R}^m$, we consider the induced distance on L^m which works with the assessments vector through the subindexes of the corresponding labels:

$$\bar{d}_p((l_{a_1}, \ldots, l_{a_m}), (l_{b_1}, \ldots, l_{b_m})) = d_p((a_1, \ldots, a_m), (b_1, \ldots, b_m)).$$

Remark 1. The ordinal scale of linguistic terms we use, L, is just a finite scale whose consecutive terms are equidistant. Following Balinski and Laraki [2], each term of the scale has associated a linguistic label. What matters is not the name of the label but the position of the label in the ordinal scale. This is the reason we consider the number of changes we need for going from a term to another one[5]. In this sense, the distance between two labels' vectors is based on the number of positions that we need to cover to go from one to another, in each of its components. To move from l_{a_i} to l_{b_i} we need to cover $|a_i - b_i|$ positions. For instance between

[4]This principle says that the relative ranking between two alternatives would only depend on the preference or assessments on these alternatives and must not be affected by other alternatives, that must be irrelevant on that comparison.

[5]This is not exactly the same that identifying each linguistic label with a number.

l_5 and l_2 we need to cover $|5-2|=3$ positions: from l_5 to l_4, from l_4 to l_3 and from l_3 to l_2.

3.2. **Election of a collective assessment for each alternative.** Our proposal is divided into several stages. First we assign a collective assessment $l(x_j) \in L$ to each alternative $x_j \in X$ which minimizes the distance between the vector of the individual assessments, $(v_j^1, \ldots, v_j^m) \in L^m$, and the vector of m replicas of the desired collective assessment, $(l(x_j), \ldots, l(x_j)) \in L^m$. For this, first we establish the set $L(x_j)$ of all the labels $l_k \in L$ satisfying

$$\bar{d}_p((v_j^1, \ldots, v_j^m), (l_k, \ldots, l_k)) \leq \bar{d}_p((v_j^1, \ldots, v_j^m), (l_h, \ldots, l_h)),$$

for each $l_h \in L$, where $(l_h, \ldots, l_h)$ and $(l_k, \ldots, l_k)$ are the vectors of m replicas of l_h and l_k, respectively. Thus, $L(x_j)$ consists of those labels that minimize the distance to the vector of individual assessments. Notice that $L(x_j) = \{l_r, \ldots, l_{r+s}\}$ is always an interval, because it contains all the terms from l_r to l_{r+s}, where $r \in \{1, \ldots, g\}$ and $0 \leq s \leq g - r$. Two different cases are possible:

(1) If $s = 0$, then $L(x_j)$ contains a single label, which will automatically be the collective assessment $l(x_j)$ of the alternative x_j.
(2) If $s > 0$, then $L(x_j)$ has more than one label. In order to select the most suitable label of $L(x_j)$, we now introduce $L^*(x_j)$, the set of all the labels $l_k \in L(x_j)$ that fulfill

$$\bar{d}_p((l_k, \ldots, l_k), (l_r, \ldots, l_{r+s})) \leq \bar{d}_p((l_h, \ldots, l_h), (l_r, \ldots, l_{r+s})),$$

for all $l_h \in L(x_j)$, where $(l_k, \ldots, l_k)$ and $(l_h, \ldots, l_h)$ are the vectors of $s+1$ replicas of l_k and l_h, respectively.
  (a) If the cardinality of $L(x_j)$ is odd, then $L^*(x_j)$ has a unique label, the median term, that will be the collective assessment $l(x_j)$.
  (b) If the cardinality of $L(x_j)$ is even, then $L^*(x_j)$ has two different labels, the two median terms. In this case, similarly to the proposal of Balinski and Laraki [2], we consider the lowest label in $L^*(x_j)$ as the collective assessment $l(x_j)$.

It is worth pointing out two different cases when we are using induced Minkowski distances.

(1) If $p = 1$, we obtain the same collective assessments that those given by MJ, the median[6] of the individual assessments. However, the final results are not necessarily the same that in MJ because we use a different tie-breaking process, as is shown later.
(2) If $p = 2$, each collective assessment is the closest label to the "mean" of the individual assessments[7], which is the one chosen by the *Range Voting* (RV) method[8] (see Smith [9]).

It is interesting to note that when we choose $p \in (1, 2)$, we find situations where the collective assessment is located between the median and the "mean". This allows

[6]It is more precise to speak about the interval of medians, because if the assessments' vector has an even number of components, then there are more than one median. See Monjardet [8].

[7]The chosen label is not exactly the arithmetic mean of the individual assessments, because we are working with a discrete spectrum of linguistic terms and not in the continuous one of the set of real numbers.

[8]RV works with a finite scale given by equidistant real numbers, and it ranks the alternatives according to the arithmetic mean of the individual assessments.

us to avoid some of the problems associated with MJ and RV. See García-Lapresta and Martínez-Panero [7] for a different proposal based on centered OWA operators (Yager [10]).

3.3. **Tie-breaking method.** Usually there exist more alternatives than linguistic terms, so it is very common to find several alternatives sharing the same collective assessment. But irrespectively of the number of alternatives, it is clear that some of them may share the same collective assessment, even when the individual assessments are very different. For these reasons it is necessary to introduce a tie-breaking method that takes into account not only the number of individual assessments above or below the obtained collective assessment (as in MJ), but the positions of these individual assessments in the ordered scale associated with L. As mentioned above, we will calculate the difference between two distances: one between $l(x_j)$ and the assessments higher than $l(x_j)$ and another one between $l(x_j)$ and the assessments lower than the $l(x_j)$. Let $\boldsymbol{v}_j^+$ and $\boldsymbol{v}_j^-$ the vectors composed by the assessments v_j^i from $\left(v_j^1, \dots, v_j^m\right)$ higher and lower than the term $l(x_j)$, respectively. First we calculate the two following distances:

$$D^+(x_j) = \bar{d}_p\left(\boldsymbol{v}_j^+, (l(x_j), \dots, l(x_j))\right),$$
$$D^-(x_j) = \bar{d}_p\left(\boldsymbol{v}_j^-, (l(x_j), \dots, l(x_j))\right),$$

where the number of components of $(l(x_j), \dots, l(x_j))$ is the same that in $\boldsymbol{v}_j^+$ and in $\boldsymbol{v}_j^-$, respectively (obviously, the number of components of $\boldsymbol{v}_j^+$ and $\boldsymbol{v}_j^-$ can be different). Once these distances have been determined, a new index $D(x_j) \in \mathbb{R}$ is calculated for each alternative $x_j \in X$: the difference between the two previous distances:

$$D(x_j) = D^+(x_j) - D^-(x_j).$$

By means of this index, we provide a kind of compensation between the individual assessments that are bigger and smaller than the collective assessment, taking into account the position of each assessment in the ordered scale associated with L. For introducing our tie-breaking process, we finally need the distance between the individual assessments and the collective one:

$$E(x_j) = \bar{d}_p\left((v_j^1, \dots, v_j^m), (l(x_j), \dots, l(x_j))\right).$$

Notice that for each alternative $x_j \in X$, $E(x_j)$ minimizes the distance between the vector of individual assessments and the linguistic labels in L, such as has been considered above in the definition of $L(x_j)$. The use of the index $E(\cdot)$ is important in the tie-breaking process because if two alternatives share the same couple $(l(\cdot), D(\cdot))$, the alternative with the lower $E(\cdot)$ is the alternative whose individual assessments are more concentrated around the collective assessment, i.e., the consensus is higher. Summarizing, for ranking the alternatives we will consider the following triplet

$$T(x_j) = (l(x_j), D(x_j), E(x_j)) \in L \times \mathbb{R} \times [0, \infty)$$

for each alternative $x_j \in X$. The sequential process works in the following way:

(1) We rank the alternatives through the collective assessments $l(\cdot)$. The alternatives with higher collective assessments will be preferred to those with lower collective assessments.

(2) If several alternatives share the same collective assessment, then we break the ties through the $D(\cdot)$ index. The alternatives with a higher $D(\cdot)$ will be preferred.
(3) If there are still ties, we break them through the $E(\cdot)$ index, in such a way such that the alternatives with a lower $E(\cdot)$ will be preferred.

Formally, the sequential process can be introduced by means of the lexicographic weak order $\succeq$ on X defined by $x_j \succeq x_k$ if and only if

(1) $l(x_j) \geq l(x_k)$ or
(2) $l(x_j) = l(x_k)$ and $D(x_j) > D(x_k)$ or
(3) $l(x_j) = l(x_k)$, $D(x_j) = D(x_k)$ and $E(x_j) \leq E(x_k)$.

Remark 2. Although it is possible that ties still exist, whenever two or more alternatives share $T(\cdot)$, these cases are very unusual when considering metrics with $p > 1$.[9] For instance, consider seven voters that assess two alternatives x_1 and x_2 by means of the set of linguistic terms given in Table 1. Table 3 includes these assessments arranged from the lowest to the highest labels. It is easy to see that

Table 3. Individual assessments

x_1	l_2	l_2	l_2	l_2	l_4	l_4	l_6
x_2	l_2	l_2	l_2	l_2	l_3	l_5	l_6

for $p = 1$ we have $T(x_1) = T(x_2) = (l_2, 8, 8)$, then $x_1 \sim x_2$ (notice that MJ and RV also provide a tie). However, if $p > 1$, the tie disappears. So, we have $x_2 \succ x_1$, excepting for $p \in (1.179, 1.203)$, where $x_1 \succ x_2$.

4. Two illustrative examples

This section focus on how the election of the parameter p is relevant in the final ranking of the alternatives. We show this fact through two different examples. The first one considers a case where the median of the individual assessments is the same for all the alternatives. And the second one considers a situation where the mean of the individual assessments' subindexes is the same for all the alternatives. In both examples we use the set of six linguistic terms $L = \{l_1, \ldots, l_6\}$ whose meaning is shown in Table 1. As mentioned above, the sequential process for ranking the alternatives is based on the triplet $T(x_j) = (l(x_j), D(x_j), E(x_j))$ for each alternative $x_j \in X$. However, by simplicity, in the following examples we only show the first two components, $(l(x_j), D(x_j))$. In these examples we also obtain the outcomes provided by MJ and RV.

Example 2. Table 4 includes the assessments given by six voters to four alternatives x_1, x_2, x_3 and x_4 arranged from the lowest to the highest labels. Notice that the mean of the individual assessments' subindexes is the same for the four alternatives, $\frac{21}{6} = 3.5$. Since RV ranks the alternatives according to this mean, it produces a tie $x_1 \sim x_2 \sim x_3 \sim x_4$. However, it is clear that this outcome might not seem reasonable, and that other rankings could be justified. Using MJ, where $l(x_1) = l(x_4) = l_4 > l_3 = l(x_2) > l_2 = l(x_3)$ and, according to the MJ

[9]The Manhattan metric ($p = 1$) produces more ties than the other metrics in the Minkowski family because of the simplicity of its calculations.

TABLE 4. Assessments in Example 2

x_1	l_1	l_2	l_4	l_4	l_4	l_6
x_2	l_1	l_1	l_3	l_4	l_6	l_6
x_3	l_2	l_2	l_2	l_4	l_5	l_6
x_4	l_1	l_1	l_4	l_5	l_5	l_5

tie-breaking process, we have $t(x_1) = -1 < 1 = t(x_4)$. Thus, MJ produces the outcome $x_4 \succ x_1 \succ x_2 \succ x_3$. We now consider the distance-based procedure for seven values of p. In Table 5 we can see the influence of these values on $(l(x_j), D(x_j))$, for $j = 1, 2, 3$. The corresponding rankings are included in Table 6. For $p = 1$, we have

TABLE 5. $(l(x_j), D(x_j))$ in Example 2

	$p = 1$	$p = 1.25$	$p = 1.5$	$p = 1.75$	$p = 2$	$p = 5$	$p = 10$
x_1	$(l_4, -3)$	$(l_4, -2.375)$	$(l_4, -2.008)$	$(l_4, -1.770)$	$(l_3, 1.228)$	$(l_3, 0.995)$	$(l_3, 0.000)$
x_2	$(l_3, 10)$	$(l_3, 2.264)$	$(l_3, 1.888)$	$(l_3, 1.669)$	$(l_3, 1.530)$	$(l_3, 1.150)$	$(l_3, 1.072)$
x_3	$(l_2, 9)$	$(l_3, 2.511)$	$(l_3, 2.254)$	$(l_3, 2.104)$	$(l_3, 2.010)$	$(l_4, -0.479)$	$(l_4, -0.232)$
x_4	$(l_4, -3)$	$(l_4, -2.815)$	$(l_4, -2.682)$	$(l_4, -2.585)$	$(l_3, 0.777)$	$(l_3, 0.199)$	$(l_3, 0.089)$

TABLE 6. Rankings in Example 2

	MJ	$p = 1$	$p = 1.25$	$p = 1.5$	$p = 1.75$	$p = 2$	$p = 5$	$p = 10$
	x_4	x_1	x_1	x_1	x_1	x_3	x_3	x_3
	x_1	x_4	x_4	x_4	x_4	x_2	x_2	x_2
	x_2	x_2	x_3	x_3	x_3	x_1	x_1	x_1
	x_3	x_3	x_2	x_2	x_2	x_4	x_4	x_4

$T(x_1) = (l_4, -3, 7)$, $T(x_2) = (l_3, 10, 11)$, $T(x_3) = (l_2, 9, 9)$ and $T(x_4) = (l_4, -3, 9)$. Then, we obtain the ranking $x_1 \succ x_4 \succ x_2 \succ x_3$, a different outcome than obtained using MJ. For $p = 1.25$, $p = 1.5$ and $p = 1.75$, we obtain $x_1 \succ x_4 \succ x_3 \succ x_2$; and for $p = 2$, $p = 5$ and $p = 10$, we have $x_3 \succ x_2 \succ x_1 \succ x_4$.

Example 3. Similarly to the previous example, Table 7 includes the assessments given by seven voters to three alternatives x_1, x_2 and x_3 arranged from the lowest to the highest labels. Clearly, the individual assessments of the three alternatives

TABLE 7. Assessments in Example 3

x_1	l_1	l_1	l_2	l_3	l_6	l_6	l_6
x_2	l_2	l_3	l_3	l_3	l_6	l_6	l_6
x_3	l_3	l_3	l_3	l_3	l_4	l_4	l_4

share the same median, l_3. According to the MJ tie-breaking process, we have

$$t(x_1) = 0 < 1 = t(x_2) = t(x_3)$$
$$N^+(x_1) = N^+(x_2) = N^+(x_3) = 3$$
$$N^-(x_3) = 0 < 1 = N^-(x_2) < 3 = N^-(x_1).$$

Thus, MJ produces the outcome $x_3 \succ x_2 \succ x_1$. This outcome does not seem logical, because x_2 has a clear advantage over x_3. On the other hand, RV ranks order the alternatives as follows: $x_2 \succ x_1 \succ x_3$, since the mean of the individual assessments' subindexes are 3.571, 4.143 and 3.429 for x_1, x_2 and x_3, respectively. We now consider the distance-based procedure for seven values of p, the same considered in the previous example. Table 8 shows the influence of these values on $(l(x_j), D(x_j))$, for $j = 1, 2, 3$. Notice that in this example the same ranking is obtained for all the

Table 8. $(l(x_j), D(x_j))$ in Example 3

	$p=1$	$p=1.25$	$p=1.5$	$p=1.75$	$p=2$	$p=5$	$p=10$
x_1	$(l_3, 4)$	$(l_3, 3.168)$	$(l_3, 2.702)$	$(l_4, -1.475)$	$(l_4, -1.332)$	$(l_4, -1.000)$	$(l_4, -0.986)$
x_2	$(l_3, 8)$	$(l_4, 0.975)$	$(l_4, 0.922)$	$(l_4, 0.868)$	$(l_4, 0.818)$	$(l_4, 0.455)$	$(l_4, 0.235)$
x_3	$(l_3, 3)$	$(l_3, 2.408)$	$(l_3, 2.080)$	$(l_3, 1.873)$	$(l_3, 1.732)$	$(l_3, 1.246)$	$(l_3, 1.116)$

considered values of p: $x_2 \succ x_1 \succ x_3$. This outcome coincides with RV, and it is more consistent than that obtained by MJ.

5. Concluding remarks

In this paper we have presented an extension of the Majority Judgement voting system developed by Balinski and Laraki [2, 3, 4, 5]. This extension is based on a distance approach but it also uses linguistic labels to evaluate the alternatives. We choose as the collective assessment for each alternative a label that minimizes the distance to the individual assessments. It is important to note that our proposal coincides in this aspect with Majority Judgement whenever the Manhattan metric is used. We also provide a tie-breaking process through the distances between the individual assessments higher and lower than the collective one. This process is richer than the one provided by Majority Judgement, which only counts the number of alternatives above or below the collective assessment, irrespectively of what they are. We also note that our tie-breaking process is essentially different to Majority Judgement even when the Manhattan metric is considered. It is important to note that using the distance-based approach we pay attention to all the individual assessments that have not been chosen as the collective assessment. With the election of a specific metric of the Minkowski family we are deciding how to evaluate these other assessments. We may distinguish four cases:

(1) If $p = 1$, the collective assessment is just the median label and no other individual assessment is relevant in this stage. However, in the tie-breaking process, all the individual assessments are taken into account, and each of them with the same weight or importance.
(2) If $p = 2$, the collective assessment is a kind of "mean" of the individual assessments because it minimizes the Euclidean distance to the individual assessments. In this stage all the voters have the same importance. However, in the tie-breaking process we are giving more importance to the assessment that are further to the collective assessment than to those labels that are closer to the collective assessment.
(3) If $p \in (1, 2)$, we are moving between the two previous cases. Higher values of p give more importance to extreme individual assessments and lower values of p give more importance to individual assessments near the median one.

(4) If $p \in (2, \infty)$, the collective assessment depends more on the extreme assessments than on the central ones, being more intense this dependency as higher the p is. If they are balanced in both sides, this has not effect in the final outcome. But if one of the sides has more extreme opinions, the collective label will go close to them. The tie-breaking process gives also more weight to the extreme opinions.

These aspects provide flexibility to our extension and it allows to devise a wide class of voting systems that may avoid some of the drawbacks related to Majority Judgement and Range Voting without losing their good features. This becomes specially interesting when the value of the parameter p in the Minkowski family belongs to the open interval $(1, 2)$, since $p = 1$ and $p = 2$ correspond to the Manhattan and the Euclidean metrics, respectively, just the metrics used in Majority Judgement and Range Voting. For instance, the election of $p = 1.5$ allows us to have a kind of compromise between both methods. As shown in the previous examples, when the value of parameter p increases, the distance-based procedure focuses more and more on the extreme assessments. However, if the individual assessments are well balanced on both sides, the outcome is not very affected by the parameter p. In further research we will analyze the properties of the presented extension of Majority Judgement within the Social Choice framework.

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